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THE LINEAR ALGEBRA OF THE PELL MATRIX

EMRAH KILIC AND DURSUN TASCI

ABSTRACT. In this paper we consider the construction of the Pell and symmetric Pell matrices. Also we discuss the linear algebra of these matrices. As applications, we derive some interesting relations involving the Pell numbers by using the properties of these Pell matrices.

1. Introduction

The Pell sequence $\{P_n\}$ is defined recursively by the equation

$$P_{n+1} = 2P_n + P_{n-1} \tag{1.1}$$

for $n \geq 2$, where $P_1 = 1$, $P_2 = 2$. The Pell sequence is

$$1, 2, 5, 12, 29, 70, 169, 408, \ldots .$$

Matrix methods are major tools in solving many problems stemming from linear recurrence relations. As is well-known (see, e.g., [1]) the numbers of this sequence are also generated by the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

since by taking successive positive powers of M one can easily establish that

$$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}.$$

In [4] and [3], the authors gave several basic Pell identities as follows, for arbitrary integers a and b,

$$P_{n+a}P_{n+b} - P_nP_{n+a+b} = P_aP_b(-1)^n, \tag{1.2}$$

$$P_{2n+1} = P_n^2 + P_{n+1}^2, \tag{1.3}$$

$$P_n = \sum_{r=0}^{[(n-1)/2]} \binom{n}{2r+1} 2^r. \tag{1.4}$$

These identities occur as Problems B-136 [8], B-155 [11] and B-161 [5], respectively.

Now we define a new matrix. The $n \times n$ *Pell matrix* $H_n = [h_{ij}]$ is defined as

$$H_n = [h_{ij}] = \begin{cases} P_{i-j+1}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0. \end{cases}$$

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For example,

$$H_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 & 0 \\ 12 & 5 & 2 & 1 & 0 & 0 \\ 29 & 12 & 5 & 2 & 1 & 0 \\ 70 & 29 & 12 & 5 & 2 & 1 \end{bmatrix},$$

and the first column of H_6 is the vector $(1, 2, 5, 12, 29, 70)^T$. Thus, the matrix H_n is useful to find the consecutive Pell numbers from the first to the nth Pell number.

The set of all n-square matrices is denoted by A_n. Any matrix $B \in A_n$ of the form $B = C^t \cdot C$, $C \in A_n$, may be written as $B = L \cdot L^t$, where $L \in A_n$ is a lower triangular matrix with nonnegative diagonal entries. This factorization is unique if C is nonsingular. This is called the *Cholesky factorization* of B. In particular, a matrix B is positive definite if and only if there exists a nonsingular lower triangular matrix $L \in A_n$ with positive diagonal entries such that $B = L \cdot L^t$. If B is a real matrix, L may be taken to be real.

A matrix $D \in A_n$ of the form

$$D = \begin{bmatrix} D_{11} & 0 & \dots & 0 \\ 0 & D_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_{kk} \end{bmatrix}$$

in which $D_{ii} \in A_{n_i}$, $i = 1, 2, \dots, k$, and $\sum_{i=1}^{k} n_i = n$, is called a *block diagonal*. Notationally, such a matrix is often indicated as $D = D_{11} \oplus D_{22} \oplus \dots \oplus D_{kk}$ or more briefly, $\oplus \sum_{i=1}^{k} D_{ii}$; this is called the *direct sum* of the matrices $D_{11}, D_{22}, \dots, D_{kk}$.

2. Pell Identities

In this section we give some identities of the Pell numbers. We start with the following Lemma.

LEMMA (2.1). *If P_n is the nth Pell number, then*

$$2P_nP_{n-1} + P_{n-1}^2 - P_n^2 = (-1)^n . \tag{2.2}$$

Proof. We will use the induction method. If $n = 1$, then we have

$$2P_1P_0 + P_0^2 - P_1^2 = -1.$$

We suppose that the equation holds for n. Now we show that the equation holds for $n + 1$. Thus

$$\begin{aligned} 2P_nP_{n-1} + P_{n-1}^2 - P_n^2 &= P_{n-1}\left(2P_n + P_{n-1}\right) - P_n^2 \\ &= \left(P_{n+1} - 2P_n\right)P_{n+1} - P_n^2 \end{aligned}$$

which, by definition of the Pell numbers, satisfy

$$\begin{aligned} 2P_nP_{n-1} + P_{n-1}^2 - P_n^2 &= -2P_nP_{n+1} - P_n^2 + P_{n+1}^2 \\ &= -\left(2P_nP_{n+1} + P_n^2 - P_{n+1}^2\right) \end{aligned}$$

which also, by induction hypothesis, satisfy

$$2P_nP_{n+1} + P_n^2 - P_{n+1}^2 = (-1)(-1)^n = (-1)^{n+1}.$$

Thus proof is complete. □

LEMMA (2.3). *Let P_n be the Pell number. Then*

$$2P_{n-1}P_n = P_{n+1}^2 - P_{n-1}^2 - 2P_nP_{n+1}.$$

Proof. By considering the proof of the previous Lemma, the proof is clear. □

LEMMA (2.4). *If P_n is the nth Pell number, then*

$$P_1^2 + P_2^2 + \ldots + P_n^2 = \frac{P_nP_{n+1}}{2}. \tag{2.5}$$

Proof. Let we take $a_i = \frac{P_iP_{i+1}}{2}$, now since

$$\begin{aligned} a_i - a_{i-1} &= \frac{P_iP_{i+1}}{2} - \frac{P_iP_{i-1}}{2} \\ &= \frac{P_i\left(P_{i+1} - P_{i-1}\right)}{2}, \end{aligned}$$

by definition of the Pell numbers, we have

$$a_i - a_{i-1} = \frac{P_i\,(2P_i)}{2} = P_i^2.$$

Now, using the idea of "creative telescoping" [13], we conclude

$$\sum_{i=2}^{n} P_i^2 = \sum_{i=2}^{n} \left(a_i - a_{i-1}\right) = a_n - a_1$$

or equivalently $(P_1 = 1)$,

$$\sum_{i=1}^{n} P_i^2 = a_n - a_1 + 1 = a_n = \frac{P_nP_{n+1}}{2}.$$

The proof is complete. □

LEMMA (2.6). *If P_n is the nth Pell number, then*

$$\begin{aligned} P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n &= \frac{P_{2n+1} - 2P_{n+1}P_n - 1}{2} \\ &= \frac{P_{2n-1} + 2P_nP_{n-1} - 1}{2}. \end{aligned} \tag{2.7}$$

Proof. From Lemma (2.3) we write the following equations for $1, 2, \ldots, n$,

$$\begin{aligned} 2P_1P_2 &= P_3^2 - P_1^2 - 2P_2P_3 \\ 2P_2P_3 &= P_4^2 - P_2^2 - 2P_3P_4 \\ 2P_3P_4 &= P_5^2 - P_3^2 - 2P_4P_5 \\ &\vdots \\ 2P_{n-2}P_{n-1} &= P_n^2 - P_{n-2}^2 - 2P_{n-1}P_n \\ 2P_{n-1}P_n &= P_{n+1}^2 - P_{n-1}^2 - 2P_nP_{n+1}. \end{aligned}$$

By addition, we obtain

$$\begin{aligned} 2\left(P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n\right) &= P_{n+1}^2 - P_{n-1}^2 - P_1^2 - P_2^2 - 2P_{n+1}P_n \\ &\quad -2\left(P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n - P_1P_2\right). \end{aligned}$$

If we arrange this equation by $P_1 = 1,\ P_2 = 5$ and equation (1.3), then we have

$$P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n = \frac{P_{2n+1} - 2P_{n+1}P_n - 1}{2}.$$

The proof is complete. □

In [2], the authors gave the Cholesky factorization of the Pascal matrix. Also in [6], the authors consider the usual Fibonacci numbers and define the Fibonacci and symmetric Fibonacci matrices. Furthermore, the authors give the factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices. In [7], the authors consider the generalized Fibobacci numbers and discuss the linear algebra of the k-Fibonacci matrix and the symmetric k-Fibonacci matrix.

3. Factorizations

In this section we consider construction and factorization of our Pell matrix of order n by using the $(0,1,2)-$matrix, where a matrix said to be a $(0,1,2)-$matrix if each of its entries are $0, 1$ or 2.

Let I_n be the identity matrix of order n. Further, we define the $n\times n$ matrices L_n, $\overline{H_n}$ and A_k by

$$L_0 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad L_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

and $L_k = L_0 \oplus I_k,\ k = 1, 2, \ldots,\ \overline{H_n} = [1] \oplus H_{n-1},\ A_1 = I_n,\ A_2 = I_{n-3} \oplus L_{-1}$, and, for $k \geq 3,\ A_k = I_{n-k} \oplus L_{k-3}$. Then we have the following Lemma.

LEMMA (3.1). $\overline{H_k} \cdot L_{k-3} = H_k,\ k \geq 3.$

Proof. For $k = 3$, we have $\overline{H_3} \cdot L_0 = H_3$. From the definition of the matrix product and familiar Pell sequence, the conclusion follows. □

Considering the previous work on Pascal functional matrices, we can rewrite $L_0,\ L_{-1}$ as follows:

$$L_{-1} = [1] \oplus P_{1,1}[1],\ L_0 = CP_{2,0}[1]\left([1] \oplus P_{1,0}[-1]\right)$$

in which $P_{n,k}[x]$ and $CP_{n,k}[x]$ are Pascal $k-$eliminated functional matrices [12].

From the definition of A_k, we know that $A_n = L_{n-3},\ A_1 = I_n$, and $A_2 = I_{n-3} \oplus L_{-1}$. The following Theorem is an immediate consequence of Lemma (3.1).

THEOREM (3.2). *The Pell matrix H_n can be factored by the A_k's as follows:*

$$H_n = A_1A_2 \ldots A_n.$$

For example

$$H_5 = A_1A_2A_3A_4A_5 = I_5(I_2 \oplus L_{-1})(I_2 \oplus L_0)([1] \oplus L_1)\, L_2$$

$$= \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&2&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&2&1&0\\0&0&1&0&1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&2&1&0&0\\0&1&0&1&0\\0&0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0&0\\2&1&0&0&0\\1&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}$$

$$= \begin{bmatrix} 1&0&0&0&0\\2&1&0&0&0\\5&2&1&0&0\\12&5&2&1&0\\29&12&5&2&1 \end{bmatrix}.$$

We give another factorization of H_n. Let $T_n = [t_{ij}]$ be $n \times n$ matrix as

$$t_{ij} = \begin{cases} P_i, & j = 1, \\ 1, & i = j, \\ 0, & \text{otherwise} \end{cases}, \quad \text{i.e.,} \quad T_n = \begin{bmatrix} P_1 & 0 & \dots & 0 \\ P_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_n & 0 & \dots & 1 \end{bmatrix}.$$

The next Theorem follows by a simple calculation.

THEOREM (3.3). *For $n \geq 2$, $H_n = T_n\left(I_1 \oplus T_{n-1}\right)\left(I_2 \oplus T_{n-2}\right)\dots\left(I_{n-2} \oplus T_2\right)$.*

We can readily find the inverse of the Pell matrix H_n. We know that

$$L_0^{-1} = \begin{bmatrix} 1&0&0\\-2&1&0\\-1&0&1 \end{bmatrix}, \quad L_{-1}^{-1} = \begin{bmatrix} 1&0&0\\0&1&0\\0&-2&1 \end{bmatrix}, \quad \text{and} \quad L_k^{-1} = L_0^{-1} \oplus I_k.$$

Define $J_k = A_k^{-1}$. Then

$$J_1 = A_1^{-1} = I_n,\ J_2 = A_2^{-1} = I_{n-3} \oplus L_1^{-1} = I_{n-2} \oplus \begin{bmatrix} 1&0\\-2&1 \end{bmatrix}, \text{ and } J_n = L_{n-3}^{-1}.$$

Also, we know that

$$T_n^{-1} = \begin{bmatrix} P_1 & 0 & 0 & \dots & 0 \\ -P_2 & 1 & 0 & \dots & 0 \\ -P_3 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -P_n & 0 & 0 & \dots & 1 \end{bmatrix} \quad \text{and} \quad \left(I_k \oplus T_{n-k}\right)^{-1} = I_k \oplus T_{n-k}^{-1}.$$

Thus the following Corollary holds.

COROLLARY (3.4).

$$H_n^{-1} = A_n^{-1}A_{n-1}^{-1}\dots A_2^{-1}A_1^{-1} = J_nJ_{n-1}\dots J_2J_1$$
$$= (I_{n-2} \oplus T_2)^{-1}\dots(I_1 \oplus T_{n-1})^{-1}T_n^{-1}.$$

From Corollary (3.4), we have

$$H_n^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -2 & 1 & 0 & 0 & \dots & 0 \\ -1 & -2 & 1 & 0 & \dots & 0 \\ 0 & -1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & \\ 0 & \dots & \dots & -1 & -2 & 1 \end{bmatrix}. \tag{3.5}$$

We define a symmetric Pell matrix $Q_n = [q_{ij}]$ as, for $i, j = 1, 2, \dots, n$,

$$q_{ij} = q_{ji} = \begin{cases} \sum\limits_{k=1}^{i} P_k^2, & i = j, \\ q_{i,j-2} + 2q_{i,j-1}, & i+1 \leq j, \end{cases}$$

in which $q_{1,0} = 0$. Then we have $q_{1j} = q_{j1} = P_j$ and $q_{2j} = q_{j2} = P_{j+1}$.
For example,

$$Q_7 = \begin{bmatrix} 1 & 2 & 5 & 12 & 29 & 70 & 169 \\ 2 & 5 & 12 & 29 & 70 & 169 & 408 \\ 5 & 12 & 30 & 72 & 174 & 420 & 1014 \\ 12 & 29 & 72 & 174 & 420 & 1014 & 2448 \\ 29 & 70 & 174 & 420 & 1015 & 2450 & 5915 \\ 70 & 169 & 420 & 1014 & 2450 & 5915 & 14280 \\ 169 & 408 & 1014 & 2448 & 5915 & 14280 & 34476 \end{bmatrix}.$$

From the definition of Q_n, we arrive at the following Lemma.

LEMMA (3.6). *For* $j \geq 3$, $q_{3j} = P_4\left(P_{j-3} + \frac{P_{j-2}P_3}{2}\right)$.

Proof. By Lemma (2.4), we have that $q_{3,3} = P_1^2 + P_2^2 + P_3^2 = \frac{P_3P_4}{2}$; hence $q_{3,3} = \frac{P_3P_4}{2} = P_4\left(P_0 + \frac{P_1P_3}{2}\right)$ for $P_0 = 0$.

By induction, $q_{3,j} = P_4\left(P_{j-3} + \frac{P_{j-2}P_3}{2}\right)$. □

We know that $q_{3,1} = q_{1,3} = P_3$ and $q_{3,2} = q_{2,3} = P_4$. Also we have that $q_{4,1} = q_{1,4}$, $q_{4,2} = q_{2,4}$ and $q_{4,3} = q_{3,4}$. By similar argument, we have the following Lemma.

LEMMA (3.7). *For* $j \geq 4$, $q_{4,j} = P_4\left(P_{j-4} + P_{j-4}P_3 + \frac{P_{j-3}P_5}{2}\right)$.

From Lemmas (3.6) and (3.7), we obtain $q_{5,1}$, $q_{5,2}$, $q_{5,3}$ and $q_{5,4}$. From these results and the definition of Q_n, we arrive at the following Lemma.

LEMMA (3.8). *For* $j \geq 5$, $q_{5,j} = P_{j-5}P_4\left(1+P_3+P_5\right)+\dfrac{P_{j-4}P_5P_6}{2}$.

Proof. Since $q_{5,5} = \dfrac{P_5P_6}{2}$ we have, by induction, $q_{5j} = P_{j-5}P_4\left(1+P_3+P_5\right)+\dfrac{P_{j-4}P_5P_6}{2}$. □

From the definition of Q_n together with Lemmas (3.6), (3.7) and (3.8) we have the following Lemma by induction on i.

LEMMA (3.9). *For* $j \geq i \geq 6$,

$$q_{ij} = P_{j-i}P_4(1+P_3+P_5)+P_{j-i}P_5P_6+P_{j-i}P_6P_7+\ldots+P_{j-i}P_{i-1}P_i+\frac{P_{j-i+1}P_iP_{i+1}}{2}.$$

Considering the above lemmas, we obtain the following result.

THEOREM (3.10). *For* $n \geq 1$ *a positive integer,* $J_nJ_{n-1}\ldots J_2J_1Q_n = H_n^T$ *and the Cholesky factorization of* Q_n *is given by* $Q_n = H_nH_n^T$.

Proof. By Corollary (3.4), $J_nJ_{n-1}\ldots J_2J_1 = H_n^{-1}$. So, if we have $H_n^{-1}Q_n = H_n^T$, then the proof is immediately seen.

Let $V = \left[v_{ij}\right] = H_n^{-1}Q_n$. Then, by (3.5), we have following:

$$v_{ij} = \begin{cases} P_j, & \text{if } i=1, \\ P_{j-1}, & \text{if } i=2, \\ -q_{i-2,j}-2q_{i-1,j}+q_{ij}, & \text{otherwise.} \end{cases}$$

Now we consider the case $i \geq 3$. Since Q_n is a symmetric matrix, $-q_{i-2,j}-2q_{i-1,j}+q_{ij} = -q_{j,i-2}-2q_{j,i-1}+q_{ji}$. Hence, by the definition of Q_n, $v_{ij} = 0$ for $j+1 \leq i$. Thus, we will prove that $-q_{i-2,j}-2q_{i-1,j}+q_{ij} = P_{j-i+1}$ for $j \geq i$. In the case in which $i \leq 5$, we have $v_{ij} = P_{j-i+1}$ by Lemmas (3.6), (3.7) and (3.8). Now we suppose that $j \geq i \geq 6$. Then by Lemma (3.9) we have

$$\begin{aligned} v_{ij} &= -q_{i-2,j}-2q_{i-1,j}+q_{ij} \\ &= (P_{j-i}-2P_{j-i+1}-P_{j-i+2})P_4(1+P_3+P_5)+(P_{j-i}-2P_{j-i+1}-P_{j-i+2})P_5P_6 \\ &\quad +\cdots+(P_{j-i}-2P_{j-i+1}-P_{j-i+2})P_{i-3}P_{i-2} \\ &\quad +\left(P_{j-i}-2P_{j-i+1}-\frac{P_{j-i+3}}{2}\right)P_{i-2}P_{i-1}+\left(P_{j-i}-P_{j-i+2}\right)P_{i-1}P_i \\ &\quad +P_{j-i+1}\frac{P_iP_{i+1}}{2}. \end{aligned}$$

Since $P_{j-i}-2P_{j-i+1}-P_{j-i+2} = -4P_{j-i+1}$, $P_{j-i}-2P_{j-i+1}-\dfrac{P_{j-i+3}}{2} = -\frac{9}{2}P_{j-i+1}$ and $P_{j-i}-P_{j-i+2} = -2P_{j-i+1}$, we obtain

$$v_{ij} = P_{j-i+1}\left[\begin{array}{l} -4P_4-4\left(P_3P_4+P_4P_5+\ldots+P_{i-3}P_{i-2}\right)- \\ \\ \frac{1}{2}P_{i-2}P_{i-1}-2P_{i-1}P_i+\frac{P_iP_{i+1}}{2}. \end{array}\right].$$

Since $P_4 = 12$, using Lemma (2.6) we get

$$\begin{aligned} v_{ij} &= P_{j-i+1} \begin{bmatrix} -48 - 4\left(\frac{P_{2(i-1)+1} - 2P_{i-1}P_i - 1}{4}\right) - 12 - \\ \frac{P_{i-2}P_{i-1}}{2} - 2P_{i-1}P_i + \frac{P_iP_{i+1}}{2} \end{bmatrix} \\ &= P_{j-i+1}\left(-P_{2i-1} + 1 - \frac{P_{i-2}P_{i-1}}{2} + \frac{P_iP_{i+1}}{2}\right). \end{aligned}$$

Using equation (1.3) and the definition of the Pell numbers we obtain

$$\begin{aligned} v_{ij} &= P_{j-i+1}\left[-2P_{i-1}^2 - 2P_i^2 + 2 - P_{i-2}P_{i-1} + P_i\left(2P_i + P_{i-1}\right)\right] \\ &= P_{j-i+1}. \end{aligned}$$

Therefore, $H_n^{-1}Q_n = H_n^T$, *i.e.*, the Cholesky factorizaton of Q_n is given by $Q_n = H_nH_n^T$. The proof is complete. □

In particular, since $Q_n^{-1} = \left(H_n^T\right)^{-1} H_n^{-1} = \left(H_n^{-1}\right)^T H_n^{-1}$, we have

$$Q_n^{-1} = \begin{bmatrix} 6 & 0 & -1 & 0 & \cdots & & \cdots & 0 \\ 0 & 6 & 0 & -1 & & & & \vdots \\ -1 & 0 & 6 & 0 & & \vdots & & \\ 0 & -1 & 0 & 6 & \cdots & & \cdots & 0 \\ \vdots & & & \vdots & & & & \vdots \\ & & & & \ddots & 6 & 0 & -1 \\ & & & & & 0 & 5 & -2 \\ 0 & \cdots & & 0 & \cdots & -1 & -2 & 1 \end{bmatrix}. \tag{3.11}$$

From Theorem (3.10), we have the following Corollary.

COROLLARY (3.12). *If P_n is the nth Pell number and k is an odd number, then*

$$P_nP_{n-k} + \ldots + P_{k+1}P_1 = \begin{cases} \left(P_nP_{n-(k-1)} - P_k\right)/2, & \textit{if n is odd}, \\ \left(P_nP_{n-(k-1)}\right)/2, & \textit{if n is even}. \end{cases}$$

If k is an even number, then

$$P_nP_{n-k} + \ldots + P_{k+1}P_1 = \begin{cases} \left(P_nP_{n-(k-1)}\right)/2, & \textit{if n is odd}, \\ \left(P_nP_{n-(k-1)} - P_k\right)/2, & \textit{if n is even}. \end{cases}$$

For the case when we multiply the ith row of H_n and the ith column of H_n^T, we obtain the formula (2.5). Also, formula (2.5) is the case when $k = 0$ in Corollary (3.12).

4. Eigenvalues of Q_n

In this section we consider the eigenvalues of Q_n.

Let $\mathfrak{B} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n;\ x_1 \geq x_2 \geq \ldots \geq x_n\}$. For $x, y \in \mathfrak{B}$, $x \prec y$ if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, $k = 1, 2, \ldots, n$ and if $k = n$, then equality holds. When $x \prec y$,

x is said to be *majorized* by y, or y is said to be *majorize* x. The condition for majorization can be written as follows: for $x, y \in \mathfrak{B}$, $x \prec y$ if $\sum_{i=0}^{k} x_{n-i} \geq \sum_{i=0}^{k} y_{n-i}$, $k = 0, 1, \ldots n-2$, and if $k = n-1$, then equality holds.

The following is an interesting simple fact:

$$(\overline{x}, \overline{x}, \ldots, \overline{x}) \prec (x_1, x_2, \ldots, x_n), \text{ where } \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

More interesting facts about majorizations can be found in [9] and [10].

An $n \times n$ matrix $P = [p_{ij}]$ is *doubly stochastic* if $p_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n$, $\sum_{i=1}^{n} P_{ij} = 1$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} P_{ij} = 1$, $i = 1, 2, \ldots, n$. In 1929, Hardy, Littlewood and Polya proved that a necessary and sufficient condition that $x \prec y$ is that there exist a doubly stochastic matrix P such that $x = yP$.

We know that both the eigenvalues and the main diagonal elements of real symmetric matrix are real numbers. The precise relationship between the main diagonal elements and the eigenvalues is given by the notion of majorization as follows: the vector of eigenvalues of a symmetric matrix is majorized by the diagonal elements of the matrix.

Note that $\det H_n = 1$ and $\det Q_n = 1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of Q_n. Since $Q_n = H_n \cdot H_n^T$ and $\sum_{i=1}^{k} P_i^2 = \frac{P_{k+1}P_k}{2}$, the eigenvalues of Q_n are all positive and

$$\left(\frac{P_{n+1}P_n}{2}, \frac{P_nP_{n-1}}{2}, \ldots, \frac{P_2P_1}{2}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

In [4], we find the combinatorial property, $P_n = \sum_{r=0}^{[(n-1)/2]} \binom{n}{2r+1} 2^r$. Therefore we have following Corollaries.

Corollary (4.1). *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of Q_n. Then*

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \begin{cases} \left[\left(\left(\sum_{r=0}^{[n/2]} \binom{n+1}{2r+1} 2^r\right)^2 - 1\right) / 4\right], & \text{if } n \text{ is odd,} \\ \left[\left(\left(\sum_{r=0}^{[n/2]} \binom{n+1}{2r+1} 2^r\right)^2\right) / 4\right], & \text{if } n \text{ is even.} \end{cases}$$

Proof. Since $\left(\frac{P_{n+1}P_n}{2}, \frac{P_nP_{n-1}}{2}, \ldots, \frac{P_2P_1}{2}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$, and from Corollary (3.12),

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \begin{cases} \frac{(P_{n+1})^2 - P_1}{4}, & \text{if } n \text{ is odd,} \\ \frac{P_{n+1}^2}{4}, & \text{if } n \text{ is even.} \end{cases}$$

By formula 1.4, the proof is immediately seen. □

COROLLARY (4.2). *If n is an odd number, then*

$$4n\lambda_n \le \left(\sum_{r=0}^{[n/2]} \binom{n+1}{2r+1} 2^r \right)^2 - 1 \le 4n\lambda_1.$$

If n is an even number, then

$$4n\lambda_n \le \left(\sum_{r=0}^{[n/2]} \binom{n+1}{2r+1} 2^r \right)^2 \le 4n\lambda_1.$$

Proof. Let $S_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. Since

$$\left(\frac{S_n}{n}, \frac{S_n}{n}, \ldots, \frac{S_n}{n} \right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

we have $\lambda_n \le \frac{S_n}{n} \le \lambda_1$. Therefore, the proof is readily seen. □

From equation (3.11), we have

$$(6, 6, \ldots, 6, 5, 1) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_1} \right). \tag{4.3}$$

Thus there exists a doubly stochastic matrix $G = [g_{ij}]$ such that

$$(6, 6, \ldots, 6, 5, 1) = \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_1} \right) \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}.$$

That is, we obtain $\frac{1}{\lambda_n} g_{1n} + \frac{1}{\lambda_{n-1}} g_{2n} + \ldots + \frac{1}{\lambda_1} g_{nn} = 1$ and $g_{1n} + g_{2n} + \ldots + g_{nn} = 1$.

LEMMA (4.4). *For each* $i = 1, 2, \ldots, n,\ g_{n-(i-1),n} \le \dfrac{\lambda_i}{n-1}$.

Proof. Suppose that $g_{n-(i-1),n} > \dfrac{\lambda_i}{n-1}$. Then

$$\begin{aligned} g_{1n} + g_{2n} + \ldots + g_{nn} &> \frac{\lambda_1}{n-1} + \frac{\lambda_2}{n-1} + \ldots + \frac{\lambda_n}{n-1} \\ &= \frac{1}{n-1} (\lambda_1 + \lambda_2 + \ldots + \lambda_n). \end{aligned}$$

Since $g_{1n} + g_{2n} + \ldots + g_{nn} = 1$ and $\sum_{i=1}^{n} \lambda_i \ge n$, this yields a contradiction, so $g_{n-(i-1),n} \le \dfrac{\lambda_i}{n-1}$. □

From Lemma (4.4), we have $1 - (n-1) \frac{1}{\lambda_i} g_{n-(i-1),n} \ge 0$. Let $\gamma = S_n - (n-1)$. Therefore, we have the following Theorem.

THEOREM (4.5). *For* $(\gamma, 1, 1, \ldots, 1) \in \mathfrak{B}$, $(\gamma, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Proof. A necessary and sufficient condition that $(\gamma, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is that there exist a doubly stochastic matrix C such that $(\gamma, 1, 1, \ldots, 1) = (\lambda_1, \lambda_2, \ldots, \lambda_n)\,C$.

We define an $n \times n$ matrix $C = [c_{ij}]$ as follows:

$$C = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{12} \\ c_{21} & c_{22} & \ldots & c_{22} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{n2} \end{bmatrix},$$

where $c_{i2} = \frac{1}{\lambda_i} g_{n-(i-1),n}$ and $c_{i1} = 1 - (n-1)\,c_{i2},\ i = 1, 2, \ldots, n$. Since G is doubly stochastic and $\lambda_i > 0$ and $c_{i2} \geq 0,\ i = 1, 2, \ldots, n$. By Lemma (4.4), $c_{i1} \geq 0,\ i = 1, 2, \ldots, n$. Then

$$c_{12} + c_{22} + \ldots + c_{n2} = \frac{g_{nn}}{\lambda_1} + \frac{g_{n-1,n}}{\lambda_2} + \ldots + \frac{g_{1n}}{\lambda_n} = 1$$

$$c_{i1} + (n-1)\,c_{i2} = 1 - (n-1)\,c_{i2} + (n-1)\,c_{i2} = 1,$$

and

$$\begin{aligned} c_{11} + c_{21} + \ldots + c_{n1} &= 1 - (n-1)\,c_{12} + 1 - (n-1)\,c_{22} + \ldots + 1 - (n-1)\,c_{n2} \\ &= n - n\,(c_{12} + c_{22} + \ldots + c_{n2}) + c_{12} + c_{22} + \ldots + c_{n2} = 1. \end{aligned}$$

Thus, G is a doubly stochastic matrix. Furthermore,

$$\begin{aligned} \lambda_1 c_{12} + \lambda_2 c_{22} + \ldots + \lambda_n c_{n2} &= \lambda_1 \frac{g_{nn}}{\lambda_1} + \lambda_2 \frac{g_{n-1,n}}{\lambda_2} + \ldots + \lambda_n \frac{g_{1n}}{\lambda_n} \\ &= g_{nn} + g_{n-1,n} + \ldots + g_{1n} = 1 \end{aligned}$$

and

$$\begin{aligned} \lambda_1 c_{11} + \lambda_2 c_{21} + \ldots + \lambda_n c_{n1} &= \lambda_1 (1 - (n-1)\,c_{12}) + \ldots + \lambda_n (1 - (n-1)\,c_{n2}) \\ &= \lambda_1 + \lambda_2 + \ldots + \lambda_n - \\ &\quad (n-1)(\lambda_1 c_{12} + \lambda_2 c_{22} + \ldots + \lambda_n c_{n2}) \\ &= \lambda_1 + \lambda_2 + \ldots + \lambda_n - (n-1) = \gamma. \end{aligned}$$

Thus, $(\gamma, 1, 1, \ldots, 1) = (\lambda_1, \lambda_2, \ldots, \lambda_n)\,C$, so $(\gamma, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$. □

From equation (4.3), we arrive at the following Lemma.

Lemma (4.6). *For* $k = 2, 3, \ldots, n,\ \lambda_k \geq \dfrac{1}{6\,(k-1)}$.

Proof. From equation (4.3), for $k \geq 2$,

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_k} \leq \underbrace{1 + 5 + 6 + + \ldots + 6}_{k} = 6(k-1).$$

Thus,

$$\frac{1}{\lambda_k} \leq 6\,(k-1) - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_{k-1}}\right) \leq 6(k-1).$$

Therefore, for $k = 2, 3, \ldots, n,\ \lambda_k \geq \dfrac{1}{6\,(k-1)}$. So the proof is complete. □

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DEPARTMENT OF MATHEMATICS
GAZI UNIVERSITY
06500 TEKNIKOKULLAR
ANKARA,
TURKEY
emkilic@gazi.edu.tr (corresponding author)
dtasci@gazi.edu.tr

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BOUNDING THE NUMBER OF SOLUTIONS OF SOME CONGRUENCES

ERIC LEVIEIL, FLORIAN LUCA, AND IGOR E. SHPARLINSKI

ABSTRACT. We estimate from above the number of solutions in integers n of congruence equations $A(n) \equiv \lambda \pmod p$, $y < n \le x$ for various sequences $A(1), \dots, A(N)$. Here p is a prime, x, y are integers, and each of our sequences under consideration is such that $a(n) = A(n)/A(n-1)$ has certain prescribed arithmetic or algebraic properties. In particular, we deal with the situations that $a(n)$ is a polynomial function of n, the n-th prime, or that a_n has some combinatorial meaning.

1. Introduction

Assume that a sequence $a(1), \dots, a(N)$ of positive rational numbers has the property that

$$A(n) = \prod_{i=1}^{n} a(i) \in \mathbb{Z} \qquad \text{for } n = 1, \dots, N.$$

Denote by $T(x, y; p, \lambda)$ the number of solutions in integers n to the congruence

$$A(n) \equiv \lambda \pmod p, \qquad y < n \le x, \tag{1.1}$$

where, throughout the paper, we always assume that p is a prime and $0 \le y < x \le N$. We estimate $T(x, y; p, \lambda)$ from above for several sequences $A(n)$. More specifically, we consider only the following sequences below, but their numerous variations can be studied as well without any substantial changes in our approach:

- products of consecutive values of a nonconstant polynomial $h(X) \in \mathbb{Z}[X]$

$$H(n) = \prod_{i=1}^{n} |h(i)|,$$

where we assume that $h(X)$ has no positive integer roots. In general, we may replace $h(X)$ by $g(X) = h(X + n_0)$, where n_0 is a positive integer which is larger than any real root of $h(X)$.

- binomial coefficients

$$C(n) = \prod_{i=1}^{n} \frac{N-i}{i};$$

- q-factorials and q-binomial coefficients

$$F_q(n) = \prod_{i=1}^{n} (q^i - 1) \qquad \text{and} \qquad C_q(n) = \prod_{i=1}^{n} \frac{q^{N-i} - 1}{q^i - 1};$$

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where $q \geq 2$ is a fixed integer;

• products of consecutive primes

$$L(n) = \prod_{i=1}^{n} \ell_i$$

where ℓ_i is the ith prime.

• products of middle binomial coefficients

$$M(n) = \prod_{i=1}^{n} \binom{2i}{i}.$$

Our approach generally follows that of [1, 3, 4], where similar questions are considered for $n!$, which corresponds to the sequence $a(i) = i$. However, treatment of each of the above sequences also requires some specific additional ingredients. To be more precise, the sequences $H(n)$, $C(n)$, $F_q(n)$ and $C_q(n)$ can be studied via just a simple variation of the arguments of [1], [3], [4]; accordingly the bounds we prove are very similar to those of these papers. On the other hand, the sequences $L(n)$ and $M(n)$ require more substantial modifications and additional arguments (and lead to bounds of different shapes).

Bounds on the number of solutions of congruences with such sequences are of independent interest. Moreover, they are important for studying other arithmetic properties of these sequences, as in [1], [2], [3], [4]; where particular instances of the results from the present paper are used to give nontrivial lower bounds on the largest prime factor of $A(n) + 1$, which hold for infinitely many values of the positive integer n.

Throughout the paper, we use the Vinogradov symbols $\gg$, $\ll$ as well as the Landau symbols O and o with their regular meanings. We recall that $U \ll V$, $V \gg U$ and $U = O(V)$ are all equivalent to the inequality $|U| \leq cV$ with some constant $c > 0$.

For $z > 0$ we use $\log z$ to denote the natural logarithm of z.

2. Products of Rational and Exponential Functions

In this section, we consider the sequences $H(n)$, $C(n)$, $F_q(n)$ and $C_q(n)$. For a prime p with $\gcd(p, q) = 1$ we denote by $\tau(p)$ the multiplicative order of q modulo p.

THEOREM (2.1). *Let $A(n)$ be one of the sequences $H(n)$, $C(n)$, $F_q(n)$ or $C_q(n)$. Then for any prime p and real x and y with $0 \leq y \leq x < p$, and also with $x \leq \tau(p)$ in the case of the sequences $F_q(n)$ or $C_q(n)$, we have*

$$\max_{\lambda=0,\ldots,p-1} T(x, y; p, \lambda) \ll (x - y)^{2/3} + 1\,.$$

Proof. Fix some integer λ, $0 \leq \lambda \leq p - 1$. Let $t = T(x, y; p, \lambda)$ and let $n_1 < \ldots < n_t$ be all the solutions to (1.1). Let $s < x$ be a positive integer. We denote by $\mathcal{U}_1$ the set of solutions n_i with either $n_i = n_t$ or $n_{i+1} \geq n_i + s$, and by $\mathcal{U}_2$ the set of all other solutions. Then

$$T(x, y; p, \lambda) = \#\mathcal{U}_1 + \#\mathcal{U}_2. \tag{2.2}$$

Obviously,

$$\#\mathcal{U}_1 \ll (x-y)/s + 1. \tag{2.3}$$

To estimate $\mathcal{U}_2$, we note that for each $n \in \mathcal{U}_2$, there is an integer k with $1 \le k < s$ such that $n+k$ is also a solution to (1.1). We consider those $n \in \mathcal{U}_2$ corresponding to a given k. For each such n we have

$$\prod_{i=1}^{k} a(n+i) \equiv 1 \pmod{p}. \tag{2.4}$$

We now consider each of the sequences $H(n)$, $C(n)$, $F_q(n)$ or $C_q(n)$ separately:

• If $a(i) = |h(i)|$ for a nonconstant polynomial $h(X) \in \mathbb{Z}[X]$, then, provided that p is large enough, the congruence (2.4) leads to two polynomial congruences modulo p (with ± 1 on the right hand side) of degree at most $k \deg h$.

• If $a(i) = (N-i)/i$, then (2.4) takes the form

$$(n+1)\dots(n+k) - (N-n-1)\dots(N-n-k) \equiv 0 \pmod{p},$$

which again leads to a nontrivial polynomial congruence of degree at most k (to see that it is nontrivial it is enough to substitute $n = -1$).

• If $a(i) = q^i - 1$, then (2.4) takes the form

$$\left(q^{n+1}-1\right)\dots\left(q^{n+k}-1\right) \equiv 1 \pmod{p},$$

which leads to a nontrivial polynomial congruence for $u = q^n$ of degree at most k. Because $0 \le x < \tau(p)$, for each value of u there is only one value of $n \in \mathcal{U}_2$.

• If $a(i) = \left(q^{N-i}-1\right) / \left(q^i - 1\right)$, then (2.4) takes the form

$$(q^{N-n-1}-1)\dots(q^{N-n-k}-1) - (q^{n+1}-1)\dots(q^{n+k}-1) \equiv 0 \pmod{p},$$

or

$$(q^{N-1}-q^n)\dots(q^{N-k}-q^n) - q^{kn}(q^{n+1}-1)\dots(q^{n+k}-1) \equiv 0 \pmod{p},$$

which leads to a nontrivial polynomial congruence for $u = q^n$ of degree at most $2k$. Because $0 \le x < \tau(p)$, for each value of u there is only one value of $n \in \mathcal{U}_2$.

Thus, in each of the above cases, (2.4) has at most $O(k)$ solutions for each fixed $k \le s$. Therefore

$$\#\mathcal{U}_2 \ll s^2. \tag{2.5}$$

Choosing $s = \left\lceil (x-y)^{1/3} \right\rceil$ to balance (2.3) and (2.5), and using (2.2), we finish the proof. □

3. Products of Primes

Here we consider only the case $y = 0$. Accordingly we define

$$T(x; p, \lambda) = T(x, 0; p, \lambda).$$

THEOREM (3.1). *Let $A(n) = L(n)$ be the product of the first n primes. Then for any prime p and real x with $0 \le x < p$, we have*

$$\max_{\lambda=0,\dots,p-1} T(x; p, \lambda) \ll x \frac{\log\log x}{\log x}.$$

Proof. As before, we fix some integer λ, $0 \le \lambda \le p-1$, and also let $n_1 < \ldots < n_t$ be all the solutions to (1.1) where $t = T(x; p, \lambda)$.

Let $s < x$ be a positive integer and let w be a real number.

We denote by $\mathcal{U}_1$ the set of solutions n_i with either $n_i = n_t$ or $n_{i+1} \ge n_i + s$, and by $\mathcal{U}_2$ the set of solutions n_i, with $\ell_{n_i+s} \ge \ell_{n_i} + ws \log x$. Finally, we denote by $\mathcal{U}_3$ the set of all other solutions. Then

$$T(x; p, \lambda) = \#\mathcal{U}_1 + \#\mathcal{U}_2 + \#\mathcal{U}_3. \tag{3.2}$$

As before, we see that $\#\mathcal{U}_1$ satisfies (2.3). By the Prime Number Theorem, we have

$$\sum_{1 \le n \le x} (\ell_{n+s} - \ell_n) \le s\ell_{\lfloor x \rfloor + s} \ll sx \log x.$$

Thus

$$\#\mathcal{U}_2 \ll x/w.$$

To estimate $\mathcal{U}_3$, we note that for each $n \in \mathcal{U}_3$, there is an integer k with $1 \le k < s$ such that $n + k$ is also a solution to (1.1). For each $n \in \mathcal{U}_3$ corresponding to a given k we have

$$\prod_{i=1}^{k} \ell_{n+i} \equiv 1 \pmod{p}.$$

Therefore, ℓ_n is a root of a polynomial congruence of the form

$$X(X + h_1) \ldots (X + h_k) \equiv 1 \pmod{p},$$

where $1 \le k \le s$ and $h_1, \ldots, h_k \in [0, ws \log x]$. Therefore, we have at most $(2ws \log x)^s$ such polynomials, and each one has at most $k \le s$ roots. Hence,

$$\#\mathcal{U}_3 \ll s(2ws \log x)^s.$$

Therefore, putting everything together, we get

$$T(x; p, \lambda) \ll x/s + x/w + s(2ws \log x)^s.$$

Choosing

$$s = w = \left\lfloor \frac{\log x}{4 \log \log x} \right\rfloor,$$

we derive the desired bound. □

4. Products of Middle Binomial Coefficients

THEOREM (4.1). *Let $A(n) = M(n)$ be the product of middle binomial coefficients. Then for any prime p and real x and y with $0 \le y \le x < p/2$, we have*

$$\max_{\lambda=0,\ldots,p-1} T(x, y; p, \lambda) \ll (x-y)^{4/5} + 1\,.$$

Proof. As before, we fix some integer λ, $0 \le \lambda \le p-1$, and also let $n_1 < \ldots < n_t$ be all the solutions to (1.1) where $t = T(x, y; p, \lambda)$. Let $s < x$ be a positive integer.

We denote by $\mathcal{U}_1$ the set of solutions n_i with either $n_i = n_t$ or $n_{i+2} \ge n_i + s$, and by $\mathcal{U}_2$ the set of all other solutions. Then

$$T(x, y; p, \lambda) = \#\mathcal{U}_1 + \#\mathcal{U}_2.$$

As before, we see that $\#\mathcal{U}_1$ satisfies (2.3).

To estimate $\#\mathcal{U}_2$, we note that for each $n \in \mathcal{U}_2$ there are integers $0 < k < m \le s$ such that $n+k$ and $n+m$ satisfy (1.1), whence

$$\prod_{j=1}^{n} \binom{2j}{j} \equiv \prod_{j=1}^{n+k} \binom{2j}{j} \equiv \prod_{j=1}^{n+m} \binom{2j}{j} \equiv \lambda \pmod p.$$

Therefore

$$\prod_{j=n+1}^{n+k} \binom{2j}{j} \equiv \prod_{j=n+1}^{n+m} \binom{2j}{j} \equiv 1 \pmod p,$$

which is equivalent to

$$2^{k(k+1)/2} \binom{2n}{n}^k \prod_{\nu=1}^{k}\prod_{i=1}^{\nu} \frac{2n+2i-1}{n+i} \equiv 2^{m(m+1)/2} \binom{2n}{n}^m \prod_{\nu=1}^{m}\prod_{i=1}^{\nu} \frac{2n+2i-1}{n+i} \equiv 1 \pmod p.$$

From this, taking into account that

$$\binom{2n}{n} \not\equiv 0 \pmod p$$

for $n \le x < p/2$, we derive

$$F_{k,m}(n) \equiv 0 \pmod p, \tag{4.2}$$

where

$$F_{k,m}(X) = 2^{km(k+1)/2} \left(\prod_{\nu=1}^{k}\prod_{i=1}^{\nu} \frac{2X+2i-1}{X+i} \right)^m - 2^{km(m+1)/2} \left(\prod_{\nu=1}^{m}\prod_{i=1}^{\nu} \frac{2X+2i-1}{X+i} \right)^k.$$

Clearly, the second term of the rational function $F_{k,m}(X)$ has a pole at $X \equiv -m \pmod p$, while the first does not (because $0 < k < m < s < x < p$). Thus, $F_{k,m}(X)$ does not vanish modulo p. Therefore, for each fixed k and m, the congruence (4.2) has at most $O(m^2)$ solutions. We also have $O(s^2)$ possible values for k, m. Therefore, the total number of such solutions n is $O(s^4)$.

This yields the bound

$$T(x, y, x, p) \ll (x-y)/s + s^4.$$

Choosing $s = \lceil (x-y)^{1/5} \rceil$, we derive $T(x, y, x, p) \ll (x-y)^{4/5} + 1$. □

5. Comments

Similar results can be obtained for many other sequences formed by sums and products of various sequences. For example, for the harmonic sums

$$S_\nu(n) = \sum_{i=1}^{n} \frac{1}{i^\nu},$$

one can easily show that the congruence

$$S_\nu(n) \equiv 0 \pmod p, \qquad y < n \le x,$$

has $O((x-y)^{2/3}+1)$ solutions for any fixed integer $\nu \neq 0$ and $x < p$ (of course only the case $\nu > 0$ is of interest, in which case all the inversions are taken modulo p). Sums of other rational functions as well as of many other sequences, including those considered in this paper, can be studied by our method as well.

We finish with posing an open question: to estimate the number of solutions of conguences with $Q(n) = \mathrm{lcm}(1, 2, \dots, n)$.

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Eric Levieil
Département d'Informatique
École Normale Supérieure
45, rue d'Ulm
75230 Paris Cedex 05
France
eric.levieil@ens.fr

Florian Luca
Instituto de Matemáticas
Universidad Nacional Autónoma de México
58089, Morelia, Michoacán
México
fluca@matmor.unam.mx

Igor E. Shparlinski
Department of Computing
Macquarie University
Sydney, NSW 2109
Australia
igor@ics.mq.edu.au

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PARTITIONS ASSOCIATED WITH GALOIS MAPS OVER p-ADIC FIELDS

ALEXANDRU ZAHARESCU

ABSTRACT. Let p be a prime number, $\mathbf{Q}_p$ the field of p-adic numbers, $\bar{\mathbf{Q}}_p$ an algebraic closure of $\mathbf{Q}_p$, and $\mathbf{C}_p$ the completion of $\bar{\mathbf{Q}}_p$ with respect to the p-adic valuation. Given a finite field extension K of $\mathbf{Q}_p$ and a map $\phi : E \to K$, with $E \subseteq \mathbf{C}_p$, we construct, via a natural Galois map, a partition of the set $\mathcal{F}_K = \{f : K \to [0, \infty]\}$. We investigate these partitions and certain regularizations associated with them.

1. Introduction

Let p be a prime number, $\mathbf{Q}_p$ the field of p-adic numbers, $\bar{\mathbf{Q}}_p$ an algebraic closure of $\mathbf{Q}_p$, and $\mathbf{C}_p$ the completion of $\bar{\mathbf{Q}}_p$ with respect to the p-adic valuation. Some metric aspects of the natural action on $\mathbf{C}_p$ of the Galois group $\mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/\mathbf{Q}_p)$ of continuous automorphisms of $\mathbf{C}_p$ over $\mathbf{Q}_p$ have been investigated, in a more general metric context, in [4], [5], [6]. In the present paper we consider further questions on this topic. We introduce and investigate certain partitions, which are defined in terms of the corresponding Galois maps, as follows.

Let K be a finite field extension of $\mathbf{Q}_p$ contained in $\bar{\mathbf{Q}}_p$, and denote $\mathcal{F}_K = \{f : K \to [0, \infty]\}$. Given a subset E of $\mathbf{C}_p$, and a map $\phi : E \to K$, consider the map $\phi^* : \mathcal{F}_K \to \mathcal{F}_E$ given by $\phi^*(f) = f \circ \phi$, for any $f \in \mathcal{F}_K$. We compose ϕ^* with the natural injection $j : \mathcal{F}_E \to \mathcal{F}_{\mathbf{C}_p}$ which sends a function $h : E \to [0, \infty]$ to the function $j(h) : \mathbf{C}_p \to [0, \infty]$ given by

$$j(h)(x) = \begin{cases} h(x), & \text{if } x \in E, \\ \infty, & \text{if } x \in \mathbf{C}_p \setminus E\,. \end{cases}$$

Next, we compose the map $j \circ \phi^* : \mathcal{F}_K \to \mathcal{F}_{\mathbf{C}_p}$ with the Galois map Gal : $\mathcal{F}_{\mathbf{C}_p} \to \mathcal{H}\left(\mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/K)\right)$, where $\mathcal{H}\left(\mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/K)\right)$ denotes the set of closed subgroups of the Galois group $\mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/K)$, and the map Gal is given by

$$\mathrm{Gal}(f) = \{\sigma \in \mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/K) : |\sigma(x) - x| \le f(x), x \in \mathbf{C}_p\},$$

for any $f \in \mathcal{F}_{\mathbf{C}_p}$. Here and in what follows $|\cdot|$ denotes the absolute value on $\mathbf{C}_p$, normalized by $|p| = \frac{1}{p}$. In this way one obtains a map $\eta : \mathcal{F}_K \to \mathcal{H}\left(\mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/K)\right)$, given by $\eta = \mathrm{Gal} \circ j \circ \phi^*$. Associated with the map η we have a partition of the set $\mathcal{F}_K$ in equivalence classes, where two elements f, g of $\mathcal{F}_K$ are equivalent if and only if $\eta(f) = \eta(g)$. In the present paper we study these partitions and certain regularizations on $\mathcal{F}_K$ which we construct in terms of such partitions. An intriguing question that arises from this investigation

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is the following. Suppose K is fixed. Suppose further that E runs over the set of finite field extensions of K, and for each such E we consider various maps $\phi : E \to K$, and look at the associated partitions they produce on the fixed set $\mathcal{F}_K$. Can one recover E from the partitions of $\mathcal{F}_K$ associated to a small set of natural maps $\phi : E \to K$, such as the trace, the norm, etc? In view of local class field theory it is conceivable that the norm alone would suffice in order to distinguish between finite abelian extensions of K via such a construction.

2. Notations, definitions and results

In this section we present some notation, definitions and results from [4], [5], and [6].

Let $(M, \leq)$ be a partially ordered set. A map $\alpha\colon M \to M$ is called an inferior regularization on M if

$$\alpha(x) \leq \alpha(y) \text{ for any } x, y \in M \text{ with } x \leq y\,,$$

$$\alpha(\alpha(x)) = \alpha(x) \text{ for any } x \in M\,,$$

$$\alpha(x) \leq x \text{ for any } x \in M\,.$$

If instead of the last condition above the map α satisfies the condition that

$$\alpha(x) \geq x \text{ for any } x \in M,$$

then one calls α a superior regularization on M. One says that x is regular with respect to α if and only if $\alpha(x) = x$.

In Proposition 2.1 from [5] it is shown that any inferior regularization on a partially ordered set is uniquely determined by its set of regular elements, and a similar result holds for superior regularizations.

We say that a partially ordered set M has inf if any subset H of M has an infimum $\inf H \in M$, that is, if for any $H \subseteq M$ there exists an element $y \in M$ such that $y \leq h$, for any $h \in H$, and if $x \in M$ is such that $x \leq h$ for any $h \in H$, then $x \leq y$. We say that an inferior or superior regularization α on M commutes with inf if for any subset H of M one has $\alpha(\inf H) = \inf \alpha(H)$.

Let E be an ultrametric space, d the distance on E and $\mathcal{F}_E = \{f : E \to [0, \infty]\}$. For any $x \in E$ and $r > 0$ denote by $B(x, r)$ the open ball of radius r centered at x.

A function $f \in \mathcal{F}_E$ is said to be metric locally constant (*m.l.c.*) provided that for any $x \in E$ and any $y \in B(x, f(x))$ one has $f(y) = f(x)$. Denote by $\widetilde{\mathcal{F}}_E$ the set of *m.l.c.* functions.

For any $z \in E$ denote by d_z the function given by $d_z(x) = d(x, z)$ for any $x \in E$. The following structure theorem is proved in Theorem 2.3 from [4]:

$\widetilde{\mathcal{F}}_E$ coincides with the smallest subset of $\mathcal{F}_E$ which contains the constants, the d_z's and is closed under taking inf and sup.

For any $f \in \mathcal{F}_E$ one defines a new element $\tilde{f} \in \mathcal{F}_E$ given by

$$\tilde{f}(x) = \inf_{y \in E} \max\{d(x, y), f(y)\}$$

for any $x \in E$. If one denotes by c_t the constant function $c_t(x) = t$, then the above relation can also be written in the form

$$\tilde{f} = \inf_{y \in E} \max\{d_y, c_{f(y)}\}.$$

The following result was obtained in [4], Theorem 2.4:

The map from $\mathcal{F}_E$ to $\mathcal{F}_E$ given by $f \mapsto \tilde{f}$ has the following properties:

(i) If $f \leq g$, then $\tilde{f} \leq \tilde{g}$.

(ii) If $\tilde{f} = g$, then $\tilde{g} = g$.

(iii) $\tilde{f} \leq f$ for any $f \in \mathcal{F}_E$.

(iv) $\tilde{\mathcal{F}}_E = \{\tilde{f} \in \mathcal{F}_E : f \in \mathcal{F}_E\} = \{f \in \mathcal{F}_E : f = \tilde{f}\}$.

(v) If H is a subset of $\mathcal{F}_E$ and $f(x) = \inf_{h\in H} h(x)$ for any $x \in E$, then $\tilde{f}(x) = \inf_{h\in H} \tilde{h}(x)$ for any $x \in E$.

It follows by the above theorem that the map $f \mapsto \tilde{f}$ is an inferior regularization on $\mathcal{F}_E$ which commutes with inf. Its set of regular elements coincides with the set $\tilde{\mathcal{F}}_E$ of metric locally constant functions.

Let now p be a prime number, and let $\mathbf{Q}_p$, $\bar{\mathbf{Q}}_p$, $\mathbf{C}_p$ and the absolute value $|\cdot|$ on $\mathbf{C}_p$ be defined as before. For any $f \in \mathcal{F}_{\mathbf{C}_p}$ consider the group

$$\mathrm{Gal}(f) = \{\sigma \in \mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/\mathbf{Q}_p) : |\sigma(x) - x| \leq f(x),\ x \in \mathbf{C}_p\}.$$

Some properties of the above map Gal are obtained in Theorem 3.3 from [5]:

(i) For any $f \in \mathcal{F}_{\mathbf{C}_p}$, $\mathrm{Gal}(f)$ is a subgroup of $\mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/\mathbf{Q}_p)$.

(ii) If $f \leq g$ then $\mathrm{Gal}(f)$ is a subgroup of $\mathrm{Gal}(g)$.

(iii) For any subset H of $\mathcal{F}_{\mathbf{C}_p}$ one has $\mathrm{Gal}(\inf_{h\in H} h) = \cap_{h\in H} \mathrm{Gal}(h)$.

(iv) $\mathrm{Gal}(\tilde{f}) = \mathrm{Gal}(f)$ for any $f \in \mathcal{F}_{\mathbf{C}_p}$.

(v) If $f, g \in \mathcal{F}_{\mathbf{C}_p}$ are such that $\tilde{f} \leq g$ and $\tilde{g} \leq f$ then $\mathrm{Gal}(f) = \mathrm{Gal}(g)$.

Let now E and F be two ultrametric spaces. We denote the distance on E and respectively on F by d_E and d_F. Any map $\varphi : E \to F$ gives rise to a map $\varphi^* : \mathcal{F}_F \to \mathcal{F}_E$ given by $\varphi^*(f) = f \circ \varphi$. We are interested in finding circumstances under which φ^* sends m.l.c. functions to m.l.c. functions, and moreover to describe the image of $\widetilde{\mathcal{F}}_F$ through φ^*.

A partial answer to the problem is provided by Proposition 1 from [6], which states that if φ is 1-Lipschitzian then $\varphi^*(\widetilde{\mathcal{F}}_F) \subseteq \widetilde{\mathcal{F}}_E$. Here, as usual, by a 1−Lipschitzian map $\varphi : E \to F$ we mean a map satisfying the inequality $d_F(\varphi(x), \varphi(y)) \leq d_E(x, y)$ for any $x, y \in E$.

One says that a map $\varphi : E \to F$ is a quasi-isometry provided that for any $x, y \in E$ one has

$$d_F(\varphi(x), \varphi(y)) = d_E(x, \varphi^{-1}(\varphi(y))).$$

Here the right hand side is defined as usual as a distance between a point and a set,

$$d_E(x, \varphi^{-1}(\varphi(y))) = \inf_{z\in\varphi^{-1}(\varphi(y))} d_E(x, z) = \inf_{\varphi(z)=\varphi(y)} d_E(x, z).$$

Note that if φ is injective then $\varphi^{-1}(\varphi(y))$ consists of y alone, thus an injective quasi-isometry is an isometry. Note also that any quasi-isometry is 1-Lipschitzian. The notion of quasi-isometry is useful in investigating the above problem. Let E, F be two ultrametric spaces as above and let $\varphi : E \to F$. If we take a function $f \in \mathcal{F}_F$, send it via φ^* to $\varphi^*(f)$ and take the regularization $\widetilde{\varphi^*(f)}$, this element of $\mathcal{F}_E$ is a m.l.c. function. If we first take the regularization of f in $\mathcal{F}_F$ and then send it via φ^* we obtain the element $\varphi^*(\tilde{f})$ of $\mathcal{F}_E$. Under the assumption that φ is 1-Lipschitzian we know from Proposition 1 from [6] that $\varphi^*(\tilde{f})$ is m.l.c., but it might not coincide with $\widetilde{\varphi^*(f)}$.

Proposition 2 from [6] states that under the stronger assumption that φ is a surjective quasi-isometry, one has the equality

$$\widetilde{\varphi^*(f)} = \varphi^*(\tilde{f}),$$

for any $f \in \mathcal{F}_F$.

We would like to have a description of the image of $\widetilde{\mathcal{F}}_F$ through φ^*. We consider the following subset $\check{\mathcal{F}}_E$ of $\mathcal{F}_E$,

$$\check{\mathcal{F}}_E = \{f \in \mathcal{F}_E : f(x) = f(y) \text{ for any } x,\, y \in E \text{ with } \varphi(x) = \varphi(y)\}.$$

Consider also the map $\varphi_* : \mathcal{F}_E \to \mathcal{F}_F$ defined by

$$\varphi_*(f)(x) = \sup_{y \in \varphi^{-1}(x)} f(y).$$

For any $f \in \mathcal{F}_E$, denote $\check{f} = \varphi^*(\varphi_*(f))$. The set $\check{\mathcal{F}}_E$ defined above may also be expressed in the form

$$\check{\mathcal{F}}_E = \{f \in \mathcal{F}_E : f = \check{f}\}.$$

A description of the image of $\widetilde{\mathcal{F}}_F$ through the map φ^* is provided by Theorem 1 from [6]:

Let E, F be ultrametric spaces and let $\varphi : E \to F$ be a surjective quasi-isometry. Then

1) The map $(\vee) : \mathcal{F}_E \to \mathcal{F}_E$ given by $f \mapsto \check{f}$ is a superior regularization on $\mathcal{F}_E$.
2) Im $\varphi^* = \check{\mathcal{F}}_E$.
3) If $f \in \check{\mathcal{F}}_E$ then $\tilde{f} \in \check{\mathcal{F}}_E$.
4) $\varphi^*(\widetilde{\mathcal{F}}_F) = \widetilde{\mathcal{F}}_E \cap \check{\mathcal{F}}_E$.

3. Partitions associated with Galois maps

In this section we work in the following context. Fix a prime number p, and let $\mathbf{Q}_p$, $\bar{\mathbf{Q}}_p$, $\mathbf{C}_p$ and the absolute value $|\cdot|$ on $\mathbf{C}_p$ be defined as before. Next, fix a finite field extension K of $\mathbf{Q}_p$, contained in $\bar{\mathbf{Q}}_p$, and consider the set $\mathcal{F}_K = \{f : K \to [0, \infty]\}$ and the Galois group $G_K := \mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/K)$.

For any subset E of $\mathbf{C}_p$, and any map $\phi : E \to K$, consider the map $\phi^* : \mathcal{F}_K \to \mathcal{F}_E$ given by $\phi^*(f) = f \circ \phi$, for any $f \in \mathcal{F}_K$. We will want to be able to employ results from [6], and for this reason we will assume in what follows that ϕ is a surjective quasi-isometry. We compose ϕ^* with the injection $j : \mathcal{F}_E \to \mathcal{F}_{\mathbf{C}_p}$ which sends each function $h : E \to [0, \infty]$ to the function $j(h) : \mathbf{C}_p \to [0, \infty]$ given by

$$j(h)(x) = \begin{cases} h(x), & \text{if } x \in E, \\ \infty, & \text{if } x \in \mathbf{C}_p \setminus E\,. \end{cases}$$

Next, we compose the map $j \circ \phi^* : \mathcal{F}_K \to \mathcal{F}_{\mathbf{C}_p}$ with the Galois map Gal : $\mathcal{F}_{\mathbf{C}_p} \to \mathcal{H}(G_K)$, where $\mathcal{H}(G_K)$ denotes the set of closed subgroups of G_K, and the map Gal is given by

$$\mathrm{Gal}(f) = \{\sigma \in G_K : |\sigma(x) - x| \le f(x), x \in \mathbf{C}_p\},$$

for any $f \in \mathcal{F}_{\mathbf{C}_p}$. In such a way we obtain a map $\eta : \mathcal{F}_K \to \mathcal{H}(G_K)$, given by $\eta = \mathrm{Gal} \circ j \circ \phi^*$. We associate to this map η a partition of $\mathcal{F}_K$ in equivalence classes, where the equivalence relation is defined as follows. Given two elements f, g of $\mathcal{F}_K$, we say that they are equivalent, and write $f \sim g$, if and only if $\eta(f) = \eta(g)$.

It is easy to see that each of the maps j, ϕ^* and Gal commutes with inf. It follows that each equivalence class above has a smallest element. As a matter of notation, for any element f of $\mathcal{F}_K$ we denote the smallest element of the equivalence class $\eta^{-1}(\eta(f))$ containing f by $\alpha(f)$. It is also easy to see that the map $\alpha : \mathcal{F}_K \to \mathcal{F}_K$ is an inferior regularization on $\mathcal{F}_K$. We remark that, although the map α might not commute with inf, its image $\operatorname{Im}\alpha$ is closed under taking inf, as we shall see below. We know by Proposition 2.1 from [5] that the equivalence relation "$\sim$" is uniquely determined by the set of α-regular elements. Let us remark that if the set E is contained in a finite field extension of K then the set of α-regular elements is finite. The following two basic questions arise. How can one compute $\alpha(f)$ for a given f? And, how can one characterize the set of α-regular elements of $\mathcal{F}_K$?

In connection with the first question above, let us take an arbitrary element f of $\mathcal{F}_K$. We look at the sequence of maps

$$\mathcal{F}_K \to \mathcal{F}_E \to \mathcal{F}_{\mathbf{C}_p} \to \mathcal{H}(G_K),$$

and the corresponding images of f through these maps,

$$f \mapsto \phi^*(f) \mapsto j(\phi^*(f)) \mapsto H := \eta(f).$$

Consider the element f_H of $\mathcal{F}_{\mathbf{C}_p}$ defined by

$$f_H(x) = \sup_{\sigma \in H} |\sigma(x) - x|, \quad x \in \mathbf{C}_p.$$

It is easy to see that f_H is the smallest element of $\mathcal{F}_{\mathbf{C}_p}$ which is sent to H by the map Gal. Note that if E does not coincide with $\mathbf{C}_p$ then f_H does not come from $\mathcal{F}_E$, in the sense that there is no element u of $\mathcal{F}_E$ for which $j(u) = f_H$. Let us denote by g the smallest element of $\mathcal{F}_E$ which is sent to H by the map $\mathrm{Gal} \circ j$. The existence of such an element g is assured by the fact that both maps j and Gal commute with inf. Then $j(g) \geq f_H$. By restricting this inequality to E we find that $g \geq f_H|_E$. This inequality holds in $\mathcal{F}_E$. Therefore $j(f_H|_E)$ lies between f_H and $j(g)$. Both f_H and $j(g)$ are sent by Gal to H. By the monotonicity of the map Gal it follows that $\mathrm{Gal}(j(f_H|_E)) = H$. Taking into account the definition of g we then obtain $f_H|_E = g$. We now make use of Theorem 1 from [6]. By the inequality $g \leq v$, where $v := \phi^*(\alpha(f)) \in \check{\mathcal{F}}_E$ we obtain

$$g \leq \check{g} \leq \check{v} = v = \phi^*(\alpha(f)),$$

and so H is also the image of $\check{g}$ in $\mathcal{H}(G_K)$. By Theorem 1 from [6] it follows that $\check{g}$ comes from $\mathcal{F}_K$, that is, there exists an element h of $\mathcal{F}_K$ for which $\check{g} = \phi^*(h)$. By the definition of $\alpha(f)$ we then have $\alpha(f) \leq h$. Next, we apply ϕ_* to the inequality $\phi^*(h) = \check{g} \leq \phi^*(\alpha(f))$ to obtain $h \leq \alpha(f)$. Therefore

$$\alpha(f) = h = \phi_*(\check{g}) = \phi_*(\phi^*(\phi_*(g))) = \phi_*(g) = \phi_*(f_H|_E).$$

This leads us to introduce the map $\theta : \mathcal{H}(G_K) \to \mathcal{F}_K$ given by

$$\theta(H) = \phi_*(f_H|_E),$$

for any $H \in \mathcal{H}(G_K)$. This map θ is useful in the problem of finding $\alpha(f)$. More precisely, by the above relations we have $\theta(H) = \alpha(f)$. Since this holds for an arbitrary element f of $\mathcal{F}_K$, we conclude that the maps η, α and θ introduced above satisfy the equality $\alpha = \theta \circ \eta$.

We now turn to our second question, which asks for a characterization of the set of α-regular elements of $\mathcal{F}_K$. Since α is an inferior regularization, we know that the set of α-regular elements of $\mathcal{F}_K$ coincides with the image $\mathrm{Im}\alpha$ of α. Therefore we look for a characterization of $\mathrm{Im}\alpha$. Let us remark at this point that by the equality $\alpha = \theta \circ \eta$ we have $\mathrm{Im}\alpha \subseteq \mathrm{Im}\,\theta$. Moreover, if the map η is surjective, then the above inclusion becomes an equality: $\mathrm{Im}\alpha = \mathrm{Im}\theta$. We will prove below that the equality $\mathrm{Im}\alpha = \mathrm{Im}\theta$ holds true regardless of whether the map η is surjective or not.

At this point we introduce a map $\beta : \mathcal{H}(G_K) \to \mathcal{H}(G_K)$, which we define as follows. Let $H \in \mathcal{H}(G_K)$. Since $\mathrm{Im}\eta$ contains the Galois group G_K, and since $\mathrm{Im}\eta$ is closed under taking inf, $\mathrm{Im}\eta$ will contain a smallest element of $\mathcal{H}(G_K)$ which contains H. We let $\beta(H)$ be this element of $\mathcal{H}(G_K)$. It is easy to see that the map β is a superior regularization on $\mathcal{H}(G_K)$. Our next goal is to show that $\beta = \eta \circ \theta$. In order to prove this equality, let H be an arbitrary element of $\mathcal{H}(G_K)$. We have the equalities

$$(\eta \circ \theta)(H) = \eta(\phi_*(f_H|_E)) = \mathrm{Gal}(j(\phi^*(\phi_*(f_H|_E)))) = \mathrm{Gal}(j(w)),$$

where $w := (\check{f_H}|_E)$. By the monotonicity of the maps Gal and j, we have the inclusion

$$\mathrm{Gal}(j(f_H|_E)) \subseteq \mathrm{Gal}(j(w)).$$

Since $j \circ \mathit{Restriction} \geq \mathit{Identity}$, we also have the inclusions

$$H \subseteq \mathrm{Gal}(f_H) \subseteq \mathrm{Gal}(j(f_H|_E)).$$

Combining the above relations we deduce that $H \subseteq (\eta \circ \theta)(H)$. Then, by the definition of $\beta(H)$ we derive that $\beta(H) \leq (\eta \circ \theta)(H)$. In conclusion, we have proved that $\beta \leq \eta\theta$. Let us recall that what we want to prove is the equality $\beta = \eta\theta$. The rest of the proof follows now from general theory of regularizations. By applying θ to the inequality $\eta \circ \theta \geq \mathit{Identity}$ we obtain

$$\theta \leq \theta\eta\theta = \alpha\theta \leq \theta,$$

where the last inequality follows from the fact that α is an inferior regularization on $\mathcal{F}_K$. Therefore one has the equality $\theta = \theta\eta\theta$. Next, by applying θ to the inequalities $\mathit{Identity} \leq \beta \leq \eta \circ \theta$ we obtain

$$\theta \leq \theta\beta \leq \theta\eta\theta = \theta.$$

It follows that $\theta = \theta\beta$. As a consequence, one also has the equality $\eta\theta = \eta\theta\beta$. Let us remark at this point that if we prove that $\eta\theta\beta \leq \beta$, then we are done, because we may combine the inequality $\eta\theta = \eta\theta\beta \leq \beta$ with the inequality $\beta \leq \eta\theta$ obtained above, in order to establish the equality $\beta = \eta\theta$.

It remains to show that $\eta\theta\beta \leq \beta$. This is a little tricky. In order to prove this inequality, the idea is to combine the equality $\theta\eta = \alpha$ which we already know, with the inequality $\alpha \leq \mathit{Identity}$ which follows from the fact that α is an inferior regularization on $\mathcal{F}_K$, and then to apply η. It follows that $\eta\theta\eta = \eta\alpha \leq \eta$. Now the point is that since $\mathrm{Im}\beta \subseteq \mathrm{Im}\eta$, from the inequality $\eta\theta\eta \leq \eta$ it follows that one also has the inequality $\eta\theta\beta \leq \beta$. This completes the proof of the desired equality $\beta = \eta\theta$.

It is now easy to obtain other equalities involving the maps $\alpha, \beta, \eta, \theta$, such as

$$\alpha\theta = \theta\eta\theta = \theta,$$
$$\theta\beta = \theta\eta\theta = \theta,$$

and

$$\beta\eta = \eta\theta\eta = \eta\alpha = \eta,$$

where the last equality follows by the definition of α.

We may state our findings in the form of a multiplication table for the maps $\alpha, \beta, \eta, \theta$. Recall that $\alpha : \mathcal{F}_K \to \mathcal{F}_K$, $\beta : \mathcal{H}(G_K) \to \mathcal{H}(G_K)$, $\eta : \mathcal{F}_K \to \mathcal{H}(G_K)$, and $\theta : \mathcal{H}(G_K) \to \mathcal{F}_K$. Thus the eight products $\alpha\beta$, $\alpha\eta$, $\beta\alpha$, $\beta\theta$, $\eta\beta$, $\eta\eta$, $\theta\alpha$, and $\theta\theta$ are not defined. For the other eight products, which are well defined, the results are collected in the following theorem.

THEOREM (3.1). *Let K be a finite field extension of $\mathbf{Q}_p$, let E be a subset of $\mathbf{C}_p$, let $\phi : E \to K$ be a surjective cuasi-isometry, and define the maps η, α, θ and β as above. Then*

$$\begin{aligned} \alpha\alpha &= \alpha, & \alpha\theta &= \theta, \\ \beta\beta &= \beta, & \beta\eta &= \eta, \\ \eta\alpha &= \eta, & \eta\theta &= \beta, \\ \theta\beta &= \theta, & \theta\eta &= \alpha. \end{aligned}$$

THEOREM (3.2). (i) *The set $\mathrm{Im}\,\alpha$ of α-regular elements of $\mathcal{F}_K$ coincides with the set $\mathrm{Im}\,\theta$.*

(ii) *The set $\mathrm{Im}\,\beta$ of β-regular elements of $\mathcal{H}(G_K)$ coincides with the set $\mathrm{Im}\,\eta$.*

(iii) *As partially ordered sets, $\mathrm{Im}\,\alpha$ and $\mathrm{Im}\,\beta$ are isomorphic. Moreover, the restrictions of η and θ to $\mathrm{Im}\,\alpha$ and respectively $\mathrm{Im}\,\beta$ are isomorphisms, inverse to each other.*

Proof of Theorem (3.2). In order to prove part (i), note first that the inclusion $\mathrm{Im}\,\alpha \subseteq \mathrm{Im}\,\theta$ clearly follows from the equality $\alpha = \theta\eta$. Similarly, the equality $\theta = \alpha\theta$ implies immediately that $\mathrm{Im}\,\theta \subseteq \mathrm{Im}\,\alpha$. By combining the above two inclusions we find that $\mathrm{Im}\,\alpha = \mathrm{Im}\,\theta$, which proves (i).

The proof of (ii) follows the same pattern. The inclusion $\mathrm{Im}\,\beta \subseteq \mathrm{Im}\,\eta$ follows from the equality $\beta = \eta\theta$, and the other inclusion, $\mathrm{Im}\,\eta \subseteq \mathrm{Im}\,\beta$, is implied by the equality $\eta = \beta\eta$.

As for part (iii), note that the restriction of η to $\mathrm{Im}\,\alpha$ sends $\mathrm{Im}\,\alpha$ to a (not necessarily strict) subset of $\mathrm{Im}\,\beta$. Indeed, if $u \in \mathrm{Im}\,\alpha$ then $u = \alpha(v)$ for some $v \in \mathcal{F}_K$, and we see that

$$\eta(u) = \eta\alpha(v) = \eta\theta\eta(v) = \beta\eta(v) \in \mathrm{Im}\,\beta.$$

Similarly, the restriction of θ to $\mathrm{Im}\,\beta$ is a map from $\mathrm{Im}\,\beta$ to $\mathrm{Im}\,\alpha$. Indeed, for any $w \in \mathrm{Im}\,\beta$, choose an element t of $\mathcal{H}(G_K)$ for which $w = \beta(t)$. Then

$$\theta(w) = \theta\beta(t) = \theta\eta\theta(t) = \alpha\theta(t) \in \mathrm{Im}\,\alpha.$$

Next, by combining the equality $\theta\eta = \alpha$ with the fact that α acts as the identity on the set $\mathrm{Im}\,\alpha$ of α-regular elements of $\mathcal{F}_K$, we deduce that the restriction of $\theta\eta$ to $\mathrm{Im}\,\alpha$ is the identity map. Similarly, the equality $\eta\theta = \beta$, together with the fact that β invariates the elements of $\mathrm{Im}\,\beta$, imply that the restriction of $\eta\theta$ to $\mathrm{Im}\,\beta$ is the identity map. In conclusion, the maps $\eta|_{\mathrm{Im}\,\alpha} : \mathrm{Im}\,\alpha \to \mathrm{Im}\,\beta$

and $\theta|_{\mathrm{Im}\,\beta} : \mathrm{Im}\,\beta \to \mathrm{Im}\,\alpha$ are inverse to each other. Taking also into account the monotonicity of these maps, we conclude that they provide isomorphisms, inverse to each other, between the partially ordered sets $\mathrm{Im}\,\alpha$ and $\mathrm{Im}\,\beta$. This completes the proof of the theorem. □

The isomorphism between $\mathrm{Im}\,\alpha$ and $\mathrm{Im}\,\beta$ from the above theorem allows one to obtain further information on the structure of these two partially ordered sets.

COROLLARY (3.3). (i) *Im α has a largest element, which is $\theta(G_K)$.*
(ii) *Im β has a smallest element, which is $\eta(0)$.*
(iii) *The sets Im α and Im β are closed under taking inf.*

Proof. Part (i) follows from the isomorphism from Theorem (3.2) (iii), together with the fact that $\mathrm{Im}\,\beta$ has a largest element, which is G_K.

Part (ii) follows similarly, taking into account that $\mathrm{Im}\,\alpha$ has a smallest element, namely the function identically zero.

As for part (iii), $\mathrm{Im}\,\alpha$ is closed under taking inf even if, as we already remarked before, α might not commute with inf. The application $H \mapsto f_H$ might not commute with inf. So η commutes with inf, but θ might not commute with inf. Now let $(H_i)_{i\in I}$ be a family of elements of $\mathrm{Im}\,\beta$, and set $H = \bigcap_{i\in I} H_i$. Here each H_i is a closed subgroup of G_K, and so H is also a closed subgroup of G_K. Since $\mathrm{Im}\,\beta = \mathrm{Im}\,\eta$, one can find for each $i \in I$ an element f_i of $\mathcal{F}_K$ for which $\eta(f_i) = H_i$. Let $f = \inf_{i\in I} f_i \in \mathcal{F}_K$. Then

$$H = \bigcap_{i\in I} H_i = \bigcap_{i\in I} \eta(f_i) = \eta(\inf_{i\in I} f_i) = \eta(f) \in \mathrm{Im}\,\eta = \mathrm{Im}\,\beta.$$

Hence $\mathrm{Im}\,\beta$ is closed under taking inf, and by the above isomorphism between $\mathrm{Im}\,\alpha$ and $\mathrm{Im}\,\beta$ it follows that $\mathrm{Im}\,\alpha$ is closed under taking inf, too. This completes the proof of the corollary. □

We conclude with some remarks on the application $H \mapsto f_H$. First, we have seen that this application is relevant to the questions discussed in this paper. More precisely, given a map $\phi : E \to K$ as above, we have constructed a partition of the set $\mathcal{F}_K$. We know that this partition is uniquely determined by the inferior regularization α, which in turn is uniquely determined by its set of regular elements. We further know that this set coincides with $\mathrm{Im}\,\alpha$, and also coincides with $\mathrm{Im}\,\theta$ by Theorem (3.2) (i). Now, in order to be able to compute $\mathrm{Im}\,\theta$ in concrete situations, one needs to be able to compute f_H for various closed subgroups H of G_K. Then, for such H, with f_H computed, and E and ϕ given, one simply restricts f_H to E and applies ϕ_* in order to find $\phi_*(f_H|_E) = \theta(H)$. As far as the actual computation of f_H is concerned, in general it is not easy to provide an exact formula for f_H. By definition, the function $f_H : \mathbf{C}_p \to [0, \infty]$ sends an element $x \in \mathbf{C}_p$ to the nonnegative real number

$$f_H(x) = \sup_{\sigma\in H} |\sigma(x) - x|.$$

If we denote by Hx the orbit of x under the action of the group H, $Hx = \{\sigma(x) : \sigma \in H\}$, then the number $f_H(x)$ may be interpreted as the diameter of Hx. Indeed, the orbit Hx is compact, and hence the function from $Hx \times Hx$ to

$[0, \infty)$ given by $(u, v) \mapsto |v - u|$ attains its maximum, at a point (x_1, x_2), say. If $\sigma_1, \sigma_2 \in H$ are such that $\sigma_1 x = x_1$, $\sigma_2 x = x_2$, and if we let $\sigma = \sigma_2^{-1}\sigma_1 \in H$, then using the fact that each element of H is an isometry on $\mathbf{C}_p$, we see that

$$|\sigma(x) - x| = |\sigma_2^{-1}\sigma_1 x - x| = |\sigma_1 x - \sigma_2 x| = |x_1 - x_2|.$$

Thus $f_H(x)$ indeed coincides with the diameter of the orbit Hx.

Another remark which is worth making is the following. By Galois theory in $\mathbf{C}_p$ (see Tate [3], Ax [1], Sen [2]) we know that the elements of $\mathcal{H}(G_K)$ are in one-to-one correspondence with the closed subfields of $\mathbf{C}_p$ which contain K. For a closed subgroup H of G_K let us denote by K_H the corresponding closed subfield of $\mathbf{C}_p$. Then, a very good approximation to f_H is provided by the distance function to K_H, which we denote by d_{K_H}. Thus d_{K_H} is defined by

$$d_{K_H}(x) = \inf\{|y - x| : y \in K_H\},$$

for any $x \in \mathbf{C}_p$. Evidently, both functions f_H and d_{K_H} vanish on K_H. Also, the functions f_H and d_{K_H} coincide on any tamely ramified extension of K_H. Moreover, the inequality

$$f_H \leq d_{K_H}$$

holds true on the entire domain $\mathbf{C}_p$. Lastly, there exists a constant $c_p > 0$ (the so called Ax-Sen constant), depending on p only, with the property that

$$f_H \geq c_p d_{K_H},$$

on the entire domain $\mathbf{C}_p$.

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Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green Street
Urbana, IL, 61801
USA
zaharesc@math.uiuc.edu

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COMPLETE INTERSECTIONS IN AFFINE MONOMIAL CURVES

ISABEL BERMEJO, PHILIPPE GIMENEZ, ENRIQUE REYES, AND RAFAEL H. VILLARREAL

ABSTRACT. Let P be the toric ideal of an affine monomial curve over an arbitrary field. Using a combinatorial-geometric approach, we characterize when P is a complete intersection in terms of certain arithmetical conditions on binary trees.

1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Given a subset I of R we denote its zero set in $\mathbb{A}_k^n$ by $V(I)$ and given a subset $X \subset \mathbb{A}_k^n$ we denote its vanishing ideal in R by $I(X)$. As usual we use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. A *binomial* in R is a difference of two monomials, that is $f = x^a - x^b$ for some $a, b \in \mathbb{N}^n$. An ideal of R generated by binomials is called a *binomial ideal*.

Let $\underline{d} = \{d_1, \ldots, d_n\}$ be a set of distinct positive integers and consider the monomial curve

$$\Gamma = \{(t^{d_1}, \ldots, t^{d_n}) \in \mathbb{A}_k^n | \, t \in k\}.$$

The homomorphism of k-algebras:

$$\phi\colon R \to k[t]; \qquad x_i \longmapsto t^{d_i}$$

is graded if we set $\deg(x_i) = d_i$ and $\deg(t) = 1$. The image of ϕ will be denoted by $k[\Gamma]$ and its kernel will be denoted by P. The ideal P is called the *toric ideal* of Γ. Since $k[t]$ is integral over $k[\Gamma]$ we have $\mathrm{ht}(P) = n - 1$. By [13], Proposition 7.1.2, the toric ideal P is generated by binomials. According to [6], Lemma 3.4, if $\gcd(\underline{d}) = 1$, Γ is an affine toric variety, that is $\Gamma = V(P)$. If k is an infinite field, we get $I(\Gamma) = P$, see [13], Corollary 7.1.12. Note that the ideal $P \subset R$ is *quasi-homogeneous*, i.e., homogeneous if one gives degree d_i to variable x_i, and one says that the *degree* of a quasi-homogeneous binomial $x^a - x^b$ in P is $a_1 d_1 + \cdots + a_n d_n$.

The prime ideal P is called a *binomial set theoretic complete intersection* if there exists a system of binomials $g_1, \ldots, g_{n-1}$ such that $P = \mathrm{rad}(g_1, \ldots, g_{n-1})$. If $P = (g_1, \ldots, g_{n-1})$ we call P a *complete intersection*. In [4] it is shown that P is generated up to radical by n binomials. In positive characteristic, P is always a binomial set theoretic complete intersection (see [10]). A clever constructive proof of this result, using diophantine equations and linear algebra, can be

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found in [1]. If k is of characteristic zero, P is a binomial set theoretic complete intersection if and only if it is a complete intersection by [2], Theorem 4. As a byproduct, we will recover this result in Section 2 (Corollary (2.6)).

There is a description of complete intersection semigroups of $\mathbb{N}$ given in [3], see also [7] for a generalization of this description to semigroups of arbitrary dimension. In the area of complete intersection toric ideals there are some recent papers; see the introduction of [11] and the references there. We present a combinatorial-geometric approach that leads to a new effective criterion for complete intersection toric ideals of affine monomial curves. This approach is different in nature to that of [3]. Using the notion of binary tree we are able to uncover a combinatorial-arithmetical structure of complete intersections. A binary tree representing a complete intersection will contain essential information of the curve Γ and its semigroup $\mathbb{N}\underline{d}$, for instance the defining equations of $k[\Gamma]$ and the Frobenius number of the numerical semigroup $\mathbb{N}\underline{d}$ (Remark (4.5)).

The contents of this paper are as follows. In Section 2, we first claim that any primary binomial ideal over a field of characteritic zero is radical (Proposition (2.3)). Its proof uses ideas introduced by Shalom Eliahou [4, 5]. Next, using a result of [6] we observe (Proposition (2.5)) that P is a complete intersection if and only if there are binomials $g_1, \ldots, g_{n-1}$ in P with $g_i = x^{\alpha_i} - x^{\beta_i}$ such that

(a) $\ker(\psi) = \mathbb{Z}\widehat{g}_1 + \cdots + \mathbb{Z}\widehat{g}_{n-1}$, where $\widehat{g}_i = \alpha_i - \beta_i$ and ψ is the linear map $\psi\colon \mathbb{Z}^n \to \mathbb{Z}$ induced by $\psi(e_i) = d_i$,

(b) $V(g_1, \ldots, g_{n-1}, x_i) = \{0\}$ for $i = 1, \ldots, n$.

For arbitrary binomials, we express the geometric condition (b) in purely combinatorial terms using the notion of "binary tree labeled by $\{1, \ldots, n\}$ and compatible with $g_1, \ldots, g_{n-1}$" (Theorem (3.7)). This result is interesting in its own right because it links geometry with discrete mathematics (digraphs) and because it can be used for arbitrary binomial ideals that need not be toric. Next, assuming that (b) holds, we characterize condition (a) in terms of arithmetical conditions on the d_i's (Proposition (4.2)). Putting it all together, we present a combinatorial-arithmetical structure theorem that characterizes when P is a complete intersection (Theorem (4.3)).

2. Binomial ideals and their radicals

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Throughout this section, I will denote a binomial ideal of R generated by $\{g_1, \ldots, g_r\}$, where $g_i = x^{\alpha_i} - x^{\beta_i}$ for $i = 1, \ldots, r$. Note that a binomial ideal does not contain any monomial of R. We denote by $\mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\}$ the subgroup of $\mathbb{Z}^n$ generated by $\widehat{g}_1 = \alpha_1 - \beta_1, \ldots, \widehat{g}_r = \alpha_r - \beta_r$. Since $\mathrm{rad}(I)$ is again a binomial ideal (see [8], Theorem 9.4 and Corollary 9.12), $\mathrm{rad}(I)$ is generated by $\{h_1, \ldots, h_s\}$ where $h_i = x^{\gamma_i} - x^{\delta_i}$ for $i = 1, \ldots, s$. If I is primary, then $h_1, \ldots, h_s$ can be chosen such that x^{γ_i} and x^{δ_i} have no common variables.

Let G be a subgroup of $\mathbb{Z}^n$. Following [4], we define an equivalence relation $\sim_G$ on the set of monomials of R by $x^\alpha \sim_G x^\beta$ if and only if $\alpha - \beta \in G$. This relation is compatible with the product. A non zero polynomial $f = \sum_\alpha \lambda_\alpha x^\alpha$ is *simple* with respect to $\sim_G$ if all its monomials with non zero coefficient are equivalent under $\sim_G$. An arbitrary non zero polynomial f in R is uniquely expressed as the sum of simple polynomials that we call its *simple components with respect to* G: $f = f_1 + \cdots + f_m$ such that f_i is simple and if $i \neq j$ and x^α, x^β are monomials in f_i and f_j respectively, then $x^\alpha \not\sim_G x^\beta$.

For convenience we recall the following result about the behaviour of simple components valid in any characteristic.

LEMMA (2.1). ([6], Lemma 2.2). *Given a non zero polynomial f in R, if $f \in I$ then any simple component of f with respect to $\mathbb{Z}\{\widehat{g}_1, \dots, \widehat{g}_r\}$ belongs to I.*

LEMMA (2.2). *If the characteristic of k is zero, then $\mathbb{Z}\{\widehat{g}_1, \dots, \widehat{g}_r\} = \mathbb{Z}\{\widehat{h}_1, \dots, \widehat{h}_s\}$.*

Proof. Set $G_1 = \mathbb{Z}\{\widehat{g}_1, \dots, \widehat{g}_r\}$ and $G_2 = \mathbb{Z}\{\widehat{h}_1, \dots, \widehat{h}_s\}$. Since $g_i \in \mathrm{rad}(I)$, then by Lemma (2.1), any simple component of g_i with respect to G_2 belongs to $\mathrm{rad}(I)$. Therefore, $\alpha_i \sim_{G_2} \beta_i$ otherwise $\mathrm{rad}(I)$ would contain x^{α_i}, which is impossible. This proves that $G_1 \subset G_2$. Observe that this holds in any characteristic.

To show the reverse containment, we adapt the argument given in the proof of [6], Proposition 2.4. Since $h_i = x^{\gamma_i} - x^{\delta_i} \in \mathrm{rad}(I)$, then $h_i^{p^m} \in I$ for $m \gg 0$ and p an arbitrary prime number. We claim that $x^{p^m\gamma_i} \sim_{G_1} x^{p^m\delta_i}$. Consider the equality

$$h_i^{p^m} = \sum_{s=0}^{p^m} (-1)^s \binom{p^m}{s} (x^{\gamma_i})^{p^m - s} (x^{\delta_i})^s.$$

If $x^{p^m\gamma_i}$ and $x^{p^m\delta_i}$ are not in the same simple component of $h_i^{p^m}$ with respect to G_1, then there is a non empty subset $S \subset \{1, \dots, p^m - 1\}$ such that the polynomial

$$f = x^{p^m\gamma_i} + \sum_{s\in S} (-1)^s \binom{p^m}{s} (x^{\gamma_i})^{p^m - s} (x^{\delta_i})^s$$

is a simple component of $h_i^{p^m}$ with respect to G_1. By Lemma (2.1), $f \in I$, and hence

$$f(1, \dots, 1) = 0 = 1 + \sum_{s\in S} (-1)^s \binom{p^m}{s},$$

a contradiction if the characteristic of k is zero because $\binom{p^m}{s} \equiv 0 \bmod (p)$ for $1 \le s \le p^m - 1$. Therefore, $x^{p^m\gamma_i} \sim_{G_1} x^{p^m\delta_i}$, and consequently $p^m(\gamma_i - \delta_i) \in G_1$. If we pick another prime number $q \neq p$ and $t \gg 0$, repeating the previous argument, we obtain $q^t(\gamma_i - \delta_i) \in G_1$, and hence $\gamma_i - \delta_i \in G_1$, as required. □

PROPOSITION (2.3). *Assume that the characteristic of k is zero. If I is primary, then* $\mathrm{rad}(I) = I$.

Proof. Let us show that $h_i = x^{\gamma_i} - x^{\delta_i}$ belongs to I for all $i = 1, \dots, s$. By Lemma (2.2), we can write

$$\gamma_i - \delta_i = \eta_1(\alpha_1 - \beta_1) + \cdots + \eta_r(\alpha_r - \beta_r) \qquad (\eta_i \in \mathbb{Z}).$$

By substituting $-g_i$ for g_i if necessary, we may assume that $\eta_1, \ldots, \eta_r \in \mathbb{N}$. Expanding the right hand side of the equality

$$\frac{h_i}{x^{\delta_i}} = \left[\left(\frac{x^{\alpha_1}}{x^{\beta_1}} - 1\right) + 1\right]^{\eta_1} \cdots \left[\left(\frac{x^{\alpha_r}}{x^{\beta_r}} - 1\right) + 1\right]^{\eta_r} - 1$$

readily gives a monomial x^γ such that $x^\gamma h_i \in I$. If $h_i \notin I$, then $(x^\gamma)^\ell \in I$ for some $\ell \geq 1$ because I is primary, but this is impossible. Thus $h_i \in I$, as required. □

Remark (2.4). Note that Proposition (2.3) fails if the characteristic of the field k is positive. For example, the primary ideal $I = (x^{10} - y^{15}) \subset \mathbb{F}_5[x, y]$ is not radical. In this example, $\mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\} \neq \mathbb{Z}\{\widehat{h}_1, \ldots, \widehat{h}_s\}$. However, one obtains as a direct consequence of the proof of Proposition (2.3) that if $\mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\} = \mathbb{Z}\{\widehat{h}_1, \ldots, \widehat{h}_s\}$ and I is primary, then $I = \mathrm{rad}(I)$. This observation is useful in the proof of our next result, which is one of the keys to our main result (Theorem (4.3)).

PROPOSITION (2.5). *Let k be an arbitrary field and let $\mathcal{B} = \{g_1, \ldots, g_{n-1}\}$ be a set of binomials in P, the toric ideal of the monomial curve Γ. Then $P = (\mathcal{B})$ if and only if*

(a) $\ker(\psi) = \mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_{n-1}\}$ *and*

(b) $V(g_1, \ldots, g_{n-1}, x_i) = \{0\}$ *for* $i = 1, \ldots, n$.

Proof. If $P = (\mathcal{B})$ then (a) follows at once from [6], Proposition 2.3, and (b) follows from [6], Theorem 3.1 (b). Conversely, if (a) and (b) hold then by [6], Theorem 3.1, one has $\mathrm{rad}(\mathcal{B}) = P$. Let $\{h_1, \ldots, h_s\}$ be a set of generators of P consisting of binomials. Notice that $\ker(\psi) = \mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\}$ by (a), and $\ker(\psi) = \mathbb{Z}\{\widehat{h}_1, \ldots, \widehat{h}_s\}$ by [6], Proposition 2.3. Thus, since $(\mathcal{B})$ is a complete intersection and its radical is a prime ideal, $(\mathcal{B})$ is radical by Remark (2.4), and hence $P = (\mathcal{B})$. □

We end this section recovering a result that holds for toric ideals of arbitrary dimension over a field of characteristic zero, see also [11], Corollary 3.10, for a recent generalization.

COROLLARY (2.6). ([2], Theorem 4). *Let $\mathfrak{p}$ be a toric ideal of R. If $\mathfrak{p}$ is a binomial set theoretic complete intersection, then $\mathfrak{p}$ is a complete intersection.*

Proof. Set $r = \dim R/\mathfrak{p}$. By hypothesis, there are $g_1, \ldots, g_{n-r}$ binomials of R such that $\mathrm{rad}(g_1, \ldots, g_{n-r}) = \mathfrak{p}$. Since the ideal $(g_1, \ldots, g_{n-r})$ is primary because it is a complete intersection and its radical is a prime ideal, the result follows from Proposition (2.3). □

3. Binary trees in binomial ideals

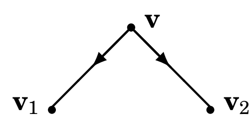
Definition (3.1). A *binary tree* is a connected directed rooted tree such that: (i) two edges leave the root and every other vertex has either degree 1 or 3, (ii) if a vertex has degree 3, then one edge enters the vertex and the other two edges leave the vertex, and (iii) if a vertex has degree 1, then one edge enters the vertex. The vertices of degree 1 are called *terminal*. For convenience we regard an isolated vertex as a binary tree.

LEMMA (3.2). *If G is a binary tree with n terminal vertices, then the number of non-terminal vertices of G is $n-1$.*

Proof. It follows by induction on n. □

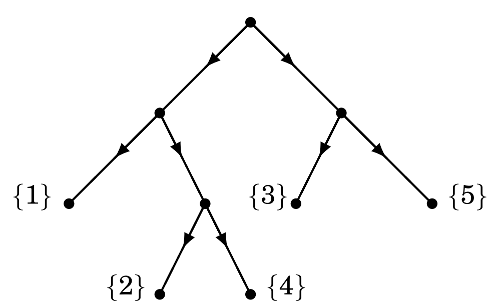
Definition (3.3). A binary tree G is said to be *labeled by* $[\![1,n]\!] := \{1,\dots,n\}$ if its terminal vertices are labeled by $\{1\},\dots,\{n\}$. Extending this definition, we will also consider binary trees with n terminal vertices labeled by arbitrary finite subsets of $\mathbb{N}$ with n elements.

If G is a binary tree labeled by $[\![1,n]\!]$ and $\mathbf{v}$ is a non-terminal vertex of G, consider $\mathbf{v}_1$ and $\mathbf{v}_2$, the two vertices of G such that



is a subgraph of G, and denote by G_1, resp. G_2, the subtree of G whose root is $\mathbf{v}_1$, resp. $\mathbf{v}_2$. We denote by $\ell_1[\mathbf{v}]$ and $\ell_2[\mathbf{v}]$ the two disjoint subsets of $[\![1,n]\!]$ formed by the union of labels of the terminal vertices of G_1 and G_2 respectively.

Example (3.4). The following binary tree is labeled by $[\![1,5]\!]$:



and if $\mathbf{v}$ is the root of G, then $\ell_1[\mathbf{v}] = \{1,2,4\}$ and $\ell_2[\mathbf{v}] = \{3,5\}$.

The *support* of a monomial x^a (resp. binomial $g = x^a - x^b$) is denoted by $\operatorname{supp}(x^a) = \{i \mid a_i > 0\}$ (resp. $\operatorname{supp}(g) = \operatorname{supp}(x^a) \cup \operatorname{supp}(x^b)$).

Definition (3.5). Let $\mathcal{B} = \{g_1,\dots,g_{n-1}\}$ be a set of binomials of R with $g_i = x^{\alpha_i} - x^{\beta_i}$, $\operatorname{supp}(x^{\alpha_i}) \cap \operatorname{supp}(x^{\beta_i}) = \emptyset$, and $\alpha_i \neq 0$, $\beta_i \neq 0$ for all $i = 1,\dots,n-1$, and let G be a binary tree labeled by $[\![1,n]\!]$. We say that G is *compatible* with $\mathcal{B}$ if, denoting by $\mathcal{F}$ the set of non-terminal vertices of G, there is a bijection

$$\mathcal{B} \xrightarrow{f} \mathcal{F}$$

such that $\operatorname{supp}(x^{\alpha_i}) \subset \ell_1[f(g_i)]$ and $\operatorname{supp}(x^{\beta_i}) \subset \ell_2[f(g_i)]$ for all $i \in \{1,\dots,n-1\}$.

Example (3.6). The binary tree G labeled by $[\![1,5]\!]$ in Example (3.4) is compatible with the set of binomials

$$\{g_1 = x_1^2x_2^4 - x_3x_5\,,\ g_2 = x_1 - x_2x_4\,,\ g_3 = x_3^4 - x_5^2\,,\ g_4 = x_2 - x_4^7\}.$$

The next result will be used later to prove our main result. It is interesting in its own right because it characterizes a geometric condition "$V(\mathcal{B}, x_i) = \{0\}$" that occurs in the study of toric curves (see [4], [6]) in terms of a combinatorial notion "labeled binary tree". In addition this result holds for arbitrary binomials not necessarily inside of a toric ideal.

THEOREM (3.7). *Let $\mathcal{B} = \{g_1, \ldots, g_{n-1}\}$ be a set of binomials of R such that $g_i = x^{\alpha_i} - x^{\beta_i}$, $\operatorname{supp}(x^{\alpha_i}) \cap \operatorname{supp}(x^{\beta_i}) = \emptyset$, and $\alpha_i \neq 0$, $\beta_i \neq 0$ for all $i = 1, \ldots, n-1$. Then the following two conditions are equivalent:*

(1) *$V(\mathcal{B}, x_i) = \{0\}$ for all $i = 1, \ldots, n$.*

(2) *There exists a binary tree G labeled by $[\![1, n]\!]$ which is compatible with $\mathcal{B}$.*

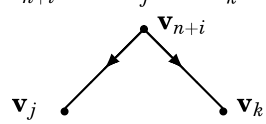
Proof. (1) $\Rightarrow$ (2): Set $V_1 := \{1\}, \ldots, V_n := \{n\}$ and consider the partition $\mathcal{F}_1 := \{V_1, \ldots, V_n\}$ of $[\![1, n]\!]$. Let us show that there exist $V_{n+1}, \ldots, V_{2n-1}$, subsets of $[\![1, n]\!]$, and $\mathcal{F}_2, \ldots, \mathcal{F}_n$, partitions of $[\![1, n]\!]$, such that, reindexing $g_1, \ldots, g_{n-1}$ if necessary, the following assertions hold for all $i \in \{1, \ldots, n-1\}$:

(a) $V_{n+i} = V_j \cup V_k$ for some $V_j, V_k \in \mathcal{F}_i$, $j \neq k$.

(b) $\operatorname{supp}(x^{\alpha_i}) \subset V_j$ and $\operatorname{supp}(x^{\beta_i}) \subset V_k$.

(c) $\mathcal{F}_{i+1} = (\mathcal{F}_i \setminus \{V_j, V_k\}) \cup \{V_{n+i}\}$.

Then, if we consider the digraph G with $2n-1$ vertices, denoted by $\mathbf{v}_1, \ldots, \mathbf{v}_{2n-1}$, where we connect $\mathbf{v}_{n+i}$ with $\mathbf{v}_j$ and $\mathbf{v}_k$ as follows:



whenever $V_{n+i} = V_j \cup V_k$ in (a), it is not hard to see that G is a binary tree labeled by $[\![1, n]\!]$. The root of G is $\mathbf{v}_{2n-1}$, and the set of its non-terminal vertices is $\mathcal{F} := \{\mathbf{v}_{n+1}, \ldots, \mathbf{v}_{2n-1}\}$. Moreover, by construction, for all $i \in \{1, \ldots, n-1\}$, one has that $\ell_1[\mathbf{v}_{n+i}] = V_j$ and $\ell_2[\mathbf{v}_{n+i}] = V_k$ for V_j and V_k in (a). Hence, by (b), G is compatible with $\mathcal{B}$ via the map $f : \mathcal{B} \to \mathcal{F}$, $g_i \mapsto \mathbf{v}_{n+i}$, and (2) will follow.

Let us first construct V_{n+1} and $\mathcal{F}_2$ satisfying (a), (b) and (c). We first claim that for all $i \in [\![1, n]\!]$, there exists an element $g_j \in \mathcal{B}$ such that either $\operatorname{supp}(x^{\alpha_j}) \subset V_i$ or $\operatorname{supp}(x^{\beta_j}) \subset V_i$ because otherwise, we have that the ith unit vector e_i of $\mathbb{A}_k^n$ belongs to $V(\mathcal{B}, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ which is $\{0\}$ by (1). Since $|\mathcal{F}_1| = n$ and $|\mathcal{B}| = n-1$, by the pigeonhole principle there exists an element in $\mathcal{B}$, say g_1, and $V_j, V_k \in \mathcal{F}_1$ with $j \neq k$, such that $\operatorname{supp}(x^{\alpha_1}) \subset V_j$ and $\operatorname{supp}(x^{\beta_1}) \subset V_k$. Setting $V_{n+1} := V_j \cup V_k$ and $\mathcal{F}_2 := (\mathcal{F}_1 \setminus \{V_j, V_k\}) \cup \{V_{n+1}\}$, the statements (a), (b) and (c) hold for $i = 1$.

Assume now that for $i \in \{2, \ldots, n-1\}$, we have constructed $V_{n+1}, \ldots, V_{n+i-1}$ and $\mathcal{F}_2, \ldots \mathcal{F}_i$ such that (a), (b) and (c) hold, and let us construct V_{n+i} and $\mathcal{F}_{i+1}$ satisfying (a), (b) and (c).

Observe first that for all $j \leq i-1$, $\operatorname{supp}(g_j)$ is contained in some element of $\mathcal{F}_i$. Set $\mathcal{B}_i := \mathcal{B} \setminus \{g_1, \ldots, g_{i-1}\}$. We claim that for each $V_k \in \mathcal{F}_i$, there exists $g_j \in \mathcal{B}_i$ such that either $\operatorname{supp}(x^{\alpha_j}) \subset V_k$ or $\operatorname{supp}(x^{\beta_j}) \subset V_k$. In order to prove this, we show that if there exists an element in $\mathcal{F}_i$, say $V_s = \{i_1, \ldots, i_m\}$, that does not satisfy the claim, then $\alpha := e_{i_1} + \cdots + e_{i_m}$ belongs to $V(\mathcal{B})$, which is a contradiction by (1). Take $g_j \in \mathcal{B}$. If $g_j \in \mathcal{B}_i$, then $\operatorname{supp}(x^{\alpha_j}) \not\subset V_s$ and $\operatorname{supp}(x^{\beta_j}) \not\subset V_s$ by definition of V_s, and hence $g_j(\alpha) = 0$. If $g_j \notin \mathcal{B}_i$, i.e., if $j \leq i-1$, then $\operatorname{supp}(g_j)$ is contained in some element of $\mathcal{F}_i$, say V_t. If $t = s$, i.e., if $\operatorname{supp}(g_j) \subset V_s$, then $g_j(\alpha) = 1 - 1 = 0$. Otherwise, since $\mathcal{F}_i$ is a partition of $[\![1, n]\!]$ and $V_s, V_t \in \mathcal{F}_i$, one has that $V_s \cap V_t = \emptyset$, and hence $\operatorname{supp}(g_j) \cap V_s = \emptyset$. Thus, $g_j(\alpha) = 0$, and the claim is proved.

We have proved that for each $V_k \in \mathcal{F}_i$, there exists $g_j \in \mathcal{B}_i$ such that either $\operatorname{supp}(x^{\alpha_j}) \subset V_k$ or $\operatorname{supp}(x^{\beta_j}) \subset V_k$. Since $|\mathcal{F}_i| = n - i + 1$ and $|\mathcal{B}_i| = n - i$, and using that $\mathcal{F}_i$ is a partition of $[\![1, n]\!]$, we get by the pigeonhole principle that there exist an element in $\mathcal{B}_i$, say g_i, and $V_j, V_k \in \mathcal{F}_i$ such that $\operatorname{supp}(x^{\alpha_i}) \subset V_j$ and $\operatorname{supp}(x^{\beta_i}) \subset V_k$. Setting $V_{n+i} := V_j \cup V_k$ and $\mathcal{F}_{i+1} = (\mathcal{F}_i \setminus \{V_j, V_k\}) \cup \{V_{n+i}\}$, the statements (a), (b) and (c) hold, and we are done.

(2) $\Rightarrow$ (1): The proof is by induction on n, the number of variables. The result is clear if $n = 2$. Denoting by $\mathbf{v}$ the root of G, we may assume without loss of generality, that $\ell_1[\mathbf{v}] = [\![1, r]\!]$ and $\ell_2[\mathbf{v}] = [\![r+1, n]\!]$ for some $r \in \{1, \ldots, n-1\}$. Then, if G_1 and G_2 are the two connected components of the digraph obtained from G by removing the vertex $\mathbf{v}$ and the two edges leaving $\mathbf{v}$, one has that G_1 and G_2 are binary trees labeled by $[\![1, r]\!]$ and $[\![r+1, n]\!]$ respectively. Reindexing the g_i's if necessary, we may also assume that G_1 is compatible with $\mathcal{B}_1 := \{g_2, \ldots, g_r\}$, G_2 is compatible with $\mathcal{B}_2 := \{g_{r+1}, \ldots, g_{n-1}\}$, and $g_1 = x^{\alpha_1} - x^{\beta_1}$ with $\operatorname{supp}(x^{\alpha_1}) \subset [\![1, r]\!]$ and $\operatorname{supp}(x^{\beta_1}) \subset [\![r+1, n]\!]$. Then, $\operatorname{supp}(g_i) \subset [\![1, r]\!]$ if $i = 2, \ldots, r$, and $\operatorname{supp}(g_i) \subset [\![r+1, n]\!]$ if $i = r+1, \ldots, n-1$. Moreover, applying the induction hypothesis, one has that $V(\mathcal{B}_1, x_i) = \{0\}$ for all $i = 1, \ldots, r$, and $V(\mathcal{B}_2, x_i) = \{0\}$ for all $i = r+1, \ldots, n$. Fix $i \in [\![1, n]\!]$ and take $a \in V(\mathcal{B}, x_i)$. The result will be proved if we show that $a = 0$. By symmetry, we may assume that $1 \leq i \leq r$. The vector $a = (a_1, \ldots, a_n)$ can be decomposed as $a = b + c$, where $b = (a_1, \ldots, a_r, 0, \ldots, 0)$. Then $b \in V(\mathcal{B}_1, x_i)$, and hence $b = 0$. On the other hand, $g_1(a) = 0$ implies that $a_j = 0$ for some $j \in \{r+1, \ldots, n\}$. Thus $c \in V(\mathcal{B}_2, x_j)$ which is $\{0\}$, and hence $a = 0$, as required. □

4. Complete intersections

Let $\underline{d} = \{d_1, \ldots, d_n\}$ be a set of distinct positive integers, and consider the monomial curve $\Gamma \subset \mathbb{A}_k^n$ and the toric ideal $P \subset k[x_1, \ldots, x_n]$ defined in the introduction. The exact sequence

$$0 \longrightarrow \ker(\psi) \longrightarrow \mathbb{Z}^n \xrightarrow{\ \psi\ } \mathbb{Z} \longrightarrow 0; \quad e_i \xmapsto{\ \psi\ } d_i$$

is related to P as follows. If $g = x^a - x^b$ is a binomial, then $g \in P$ if and only if $a - b \in \ker(\psi)$.

Given a binomial $g = x^a - x^b$, we set $\widehat{g} = a - b$. If $\alpha = (\alpha_i) \in \mathbb{Z}^n$, its support is given by $\operatorname{supp}(\alpha) = \{i \mid \alpha_i \neq 0\}$. Any $\alpha \in \mathbb{Z}^n$ can be written as $\alpha = \alpha^+ - \alpha^-$, where α^+ and α^- are vectors in $\mathbb{N}^n$ with disjoint support. If $S \subset \mathbb{N}^n$, the subsemigroup (resp. subgroup) of $\mathbb{N}^n$ (resp. $\mathbb{Z}^n$) generated by S will be denoted by $\mathbb{N}S$ (resp. $\mathbb{Z}S$).

Definition (4.1). Let G be a binary tree labeled by $[\![1, n]\!]$, and consider a set of vectors in $\mathbb{Z}^n$, $W = \{w_1, \ldots, w_{n-1}\}$. We say that G is *compatible* with W if G is compatible with the set of binomials $\{x^{w_i^+} - x^{w_i^-}\,;\ i = 1, \ldots, n-1\}$.

PROPOSITION (4.2). *Let G be a binary tree labeled by $[\![1,n]\!]$ and denote by $\mathcal{F}$ the set of its non-terminal vertices. The following two conditions are equivalent*:

(1) *There exist vectors $w_1,\ldots,w_{n-1}\in\mathbb{Z}^n$ such that G is compatible with $W=\{w_1,\ldots,w_{n-1}\}$, and* $\ker(\psi)=\mathbb{Z}W$.

(2) *For all* $\mathbf{v}\in\mathcal{F}$,

$$\frac{\gcd(d_j,\ j\in\ell_1[\mathbf{v}])\gcd(d_j,\ j\in\ell_2[\mathbf{v}])}{\gcd(d_j,\ j\in\ell_1[\mathbf{v}]\cup\ell_2[\mathbf{v}])}\in\mathbb{N}\{d_j,\ j\in\ell_1[\mathbf{v}]\}\cap\mathbb{N}\{d_j,\ j\in\ell_2[\mathbf{v}]\}.$$

Proof. Let $\mathbf{v}$ be the root of G, and consider G_1 and G_2, the two components of the digraph obtained from G by removing the vertex $\mathbf{v}$ and the two edges leaving $\mathbf{v}$. We may assume that $\ell_1[\mathbf{v}]=[\![1,r]\!]$ and $\ell_2[\mathbf{v}]=[\![r+1,n]\!]$ for some $1\le r\le n-1$. Then G_1 and G_2 are binary trees labeled by $[\![1,r]\!]$ and $[\![r+1,n]\!]$. The result is clear if $n=2$. We will prove both implications by induction on n.

(1) $\Rightarrow$ (2): Reindexing the w_i's if necessary, we may assume that w_{n-1} is the element of W associated to $\mathbf{v}$ through the map that makes G compatible with W, and that $W_1=\{w_1,\ldots,w_{r-1}\}$ and $W_2=\{w_r,\ldots,w_{n-2}\}$ are the set of vectors in W such that G_i is compatible with W_i for $i=1,2$. There is a decomposition $\mathbb{Z}^n=\mathbb{Z}^r\oplus\mathbb{Z}^{n-r}$, where $\mathbb{Z}^r:=\mathbb{Z}^r\times\{0\}^{n-r}$ and $\mathbb{Z}^{n-r}:=\{0\}^r\times\mathbb{Z}^{n-r}$. Consider the linear map $\overline{\psi}_1\colon\mathbb{Z}^n\to\mathbb{Z}$ induced by $\overline{\psi}_1(e_i)=d_i$ if $1\le i\le r$ and $\overline{\psi}_1(e_i)=0$ if $r<i\le n$, and the map $\overline{\psi}_2=\psi-\overline{\psi}_1$. Let ψ_1 (resp. ψ_2) be the restriction of $\overline{\psi}_1$ (resp. $\overline{\psi}_2$) to $\mathbb{Z}^r$ (resp. $\mathbb{Z}^{n-r}$). We claim that $\ker(\psi_1)=\mathbb{Z}W_1$ and $\ker(\psi_2)=\mathbb{Z}W_2$. By symmetry it suffices to prove the first equality. Clearly one has $\mathbb{Z}W_1\subset\ker(\psi_1)$ because $\mathrm{supp}(w_i)\subset[\![1,r]\!]$ for $1\le i\le r$. To show the reverse inclusion take $\alpha\in\ker(\psi_1)\subset\mathbb{Z}^r$. Since $\alpha\in\ker(\psi)=\mathbb{Z}W$ we can write

$$\alpha=(\lambda_1w_1+\cdots+\lambda_{r-1}w_{r-1})+(\lambda_rw_r+\cdots+\lambda_{n-2}w_{n-2})+\lambda_{n-1}w_{n-1}\ (\lambda_i\in\mathbb{Z}).$$

Hence $0=\psi_1(\alpha)=\overline{\psi}_1(\alpha)=\lambda_{n-1}\overline{\psi}_1(w_{n-1})=\lambda_{n-1}\overline{\psi}_1(w^+_{n-1})$. In the last equality we use $\mathrm{supp}(w^+_{n-1})\subset[\![1,r]\!]$ and $\mathrm{supp}(w^-_{n-1})\subset[\![r+1,n]\!]$. As $\overline{\psi}_1(w^+_{n-1})\neq0$ we get $\lambda_{n-1}=0$. Therefore $\lambda_rw_r+\cdots+\lambda_{n-2}w_{n-2}=0$. This prove that $\alpha\in\mathbb{Z}W_1$, as required. Set

$$\begin{aligned}
d&=\gcd(d_1,\ldots,d_n), & d'&=\gcd(d_1,\ldots,d_r), & d''&=\gcd(d_{r+1},\ldots,d_n),\\
\underline{d}&=\{d_1,\ldots,d_n\}, & \underline{d}'&=\{d_1,\ldots,d_r\}, & \underline{d}''&=\{d_{r+1},\ldots,d_n\}.
\end{aligned}$$

Using induction and the claim we need only show $(d'd'')/d\in\mathbb{N}\underline{d}'\cap\mathbb{N}\underline{d}''$. For $1\le j\le r$ and $r+1\le k\le n$ we can write

$$\frac{d_k}{d}e_j-\frac{d_j}{d}e_k=\lambda^1_{jk}w_1+\cdots+\lambda^{r-1}_{jk}w_{r-1}+\lambda^r_{jk}w_r+\cdots+\lambda^{n-2}_{jk}w_{n-2}+\lambda^{n-1}_{jk}w_{n-1},$$

for some $\lambda^1_{kj},\ldots,\lambda^{n-1}_{kj}$ in $\mathbb{Z}$. We write the last vector in W as $w_{n-1}=w^+_{n-1}-w^-_{n-1}$ and $w_{n-1}=(a_1,\ldots,a_r,-a_{r+1},\ldots,-a_n)$. Hence we get

$$\begin{aligned}
-(d_je_k)/d&=\lambda^r_{jk}w_r+\cdots+\lambda^{n-2}_{jk}w_{n-2}-\lambda^{n-1}_{jk}w^-_{n-1}\ \Rightarrow\\
(d_jd_k)/d&=\lambda^{n-1}_{jk}(a_{r+1}d_{r+1}+\cdots+a_nd_n).
\end{aligned}$$

Set $h=a_{r+1}d_{r+1}+\cdots+a_nd_n$. If we fix k and vary j, we get

$$\gcd\big((d_1d_k/d),\ldots,(d_rd_k)/d\big)=\mu_kh\quad(\mu_k\in\mathbb{Z})\ \Rightarrow\ d_kd'=\mu_khd.$$

Therefore varying k yields $\gcd\left(d_{r+1}d',\ldots,d_nd'\right) = hd$, $\mu\in\mathbb{Z}$. As a consequence $(d'd'')/d = (hd\mu)/d \in \mathbb{N}\underline{d}''$. A symmetric argument gives $(d'd'')/d \in \mathbb{N}\underline{d}'$, as required.

(2) ⇒ (1): By induction there are $W_1=\{w_1,\ldots,w_{r-1}\}$, $W_2=\{w_r,\ldots,w_{n-2}\}$ such that G_i is compatible with W_i and $\ker(\psi_i)=\mathbb{Z}W_i$. The result will be proved if we give $w_{n-1}\in\mathbb{Z}^n$ such that $\operatorname{supp}(w_{n-1}^+)\subset[\![1,r]\!]$, $\operatorname{supp}(w_{n-1}^-)\subset[\![r+1,n]\!]$, and $\ker(\psi)=\mathbb{Z}W$ for $W=W_1\cup W_2\cup\{w_{n-1}\}$. By hypothesis,

$$(d'd'')/d = a_1d_1+\cdots+a_rd_r = a_{r+1}d_{r+1}+\cdots+a_nd_n,$$

where $a_i\in\mathbb{N}$ for all i. Setting $w_{n-1} := (a_1,\ldots,a_r,-a_{r+1},\ldots,-a_n)$, one has that $\operatorname{supp}(w_{n-1}^+)\subset[\![1,r]\!]$ and $\operatorname{supp}(w_{n-1}^-)\subset[\![r+1,n]\!]$, and hence G is compatible with $W := W_1\cup W_2\cup\{w_{n-1}\}$. To complete the proof it remains to prove the equality $\mathbb{Z}W=\ker(\psi)$. Clearly $\mathbb{Z}W\subset\ker(\psi)$. To prove the reverse containment define $\sigma_{jk} = (d_j/d)e_k-(d_k/d)e_j$, $j,k\in[\![1,n]\!]$. By [13], Corollary 10.1.10, the set $\{\sigma_{jk}\,|\, j,k\in[\![1,n]\!]\}$ generates $\ker(\psi)$. Thus we need only show that $\sigma_{jk}\in\mathbb{Z}W$ for all $j,k\in[\![1,n]\!]$. If $j,k\in[\![1,r]\!]$ or $j,k\in[\![r+1,n]\!]$, then $\sigma_{jk}\in\ker(\psi_1)\subset\mathbb{Z}W$ or $\sigma_{jk}\in\ker(\psi_2)\subset\mathbb{Z}W$. Assume $j\in[\![1,r]\!]$ and $k\in[\![r+1,n]\!]$. From the equalities

$$\begin{aligned} S_1 &= \sum_{i=1}^{r} a_i\left((d_i/d')e_j-(d_j/d')e_i\right) = (d''/d)e_j-(d_j/d')\sum_{i=1}^{r}a_ie_i,\\ S_2 &= \sum_{i=r+1}^{n} a_i\left((d_i/d'')e_k-(d_k/d'')e_i\right) = (d'/d)e_k-(d_k/d'')\sum_{i=r+1}^{n}a_ie_i \end{aligned}$$

we conclude

$$(d_k/d'')S_1-(d_j/d')S_2 = \left((d_k/d)e_j-(d_j/d)e_k\right)-(d_jd_k/d'd'')w_{n-1}.$$

Since $S_i\in\ker(\psi_i)\subset\mathbb{Z}W$ we obtain $\sigma_{jk}\in\mathbb{Z}W$, as required. □

THEOREM (4.3). *The toric ideal P is a complete intersection if and only if there is a binary tree G labeled by $[\![1,n]\!]$ such that, for all non-terminal vertex $\mathbf{v}$ of G, one has that*

$$\frac{\gcd(d_j,\ j\in\ell_1[\mathbf{v}])\gcd(d_j,\ j\in\ell_2[\mathbf{v}])}{\gcd(d_j,\ j\in\ell_1[\mathbf{v}]\cup\ell_2[\mathbf{v}])}\in\mathbb{N}\{d_j,\ j\in\ell_1[\mathbf{v}]\}\cap\mathbb{N}\{d_j,\ j\in\ell_2[\mathbf{v}]\}.$$

Proof. ⇒) There are binomials $g_1,\ldots,g_{n-1}$ such that $P=(g_1,\ldots,g_{n-1})$. We may assume that $g_i=x^{\alpha_i}-x^{\beta_i}$ and $\operatorname{supp}(x^{\alpha_i})\cap\operatorname{supp}(x^{\beta_i})=\emptyset$ for all i. By Proposition (2.5) (b) and Theorem (3.7) there exists a binary tree G labeled by $[\![1,n]\!]$ which is compatible with $\{g_1,\ldots,g_{n-1}\}$. Then G is compatible with $W=\{\widehat{g}_1,\ldots,\widehat{g}_{n-1}\}$ and $\ker(\psi)=\mathbb{Z}\{\widehat{g}_1,\ldots,\widehat{g}_{n-1}\}$ (see Proposition (2.5) (a)). Thus applying Proposition (4.2) we obtain the required conditions.

⇐) By Proposition (4.2) there is $W=\{w_1,\ldots,w_{n-1}\}\subset\mathbb{Z}^n$ such that W is compatible with G and $\ker(\psi)=\mathbb{Z}W$. Setting $g_i := x^{w_i^+}-x^{w_i^-}$, one has that G is compatible with $\{g_1,\ldots,g_{n-1}\}$, and hence, using Theorem (3.7), we get

$$V(g_1,\ldots,g_{n-1},x_i)=\{0\}\quad (i=1,\ldots,n).$$

Therefore by Proposition (2.5) we deduce the equality $P=(g_1,\ldots,g_{n-1})$. □

Using a different approach, Delorme characterizes in [3] toric ideals of affine monomial curves that are complete intersections using a tool that he calls *suites distinguées* ([3], Lemme 8). He then deduces his main result that can also be obtained from our characterization in terms of binary trees:

COROLLARY (4.4). ([3], Proposition 9). *Assume that* $\gcd(\underline{d}) = 1$. *Then P is a complete intersection if and only if, reindexing the d_i's if necessary, there exists $r \in \{1,\dots,n-1\}$ such that, setting $d' := \gcd(d_1,\dots,d_r)$, $d'' := \gcd(d_{r+1},\dots,d_n)$, and*

$$d'_i := \begin{cases} \frac{d_i}{d'} & \text{if } 1 \le i \le r \\ \frac{d_i}{d''} & \text{if } r+1 \le i \le n \end{cases},$$

one has that:

(a) $d' \in \mathbb{N}\{d'_{r+1},\dots,d'_n\}$, $d'' \in \mathbb{N}\{d'_1,\dots,d'_r\}$, *and*

(b) *the two toric ideals $P_1 \subset k[x_1,\dots,x_r]$ and $P_2 \subset k[x_{r+1},\dots,x_n]$ defined by $\{d_1,\dots,d_r\}$ and $\{d_{r+1},\dots,d_n\}$ respectively, are both complete intersections.*

Proof. $\Rightarrow$) If P is a complete intersection and $\mathbf{v}$ is the root of the binary tree G given by Theorem (4.3), we may assume, reindexing the d_i's if necessary, that $\ell_1[\mathbf{v}] = [\![1,r]\!]$ and $\ell_2[\mathbf{v}] = [\![r+1,n]\!]$ for some $r \in \{1,\dots,n-1\}$. Setting $d' := \gcd(d_1,\dots,d_r)$ and $d'' := \gcd(d_{r+1},\dots,d_n)$, one gets that $d'd'' \in \mathbb{N}\{d_1,\dots,d_r\} \cap \mathbb{N}\{d_{r+1},\dots,d_n\}$ by Theorem (4.3), and (a) follows. Moreover, using the two binary subtrees of G obtained by removing $\mathbf{v}$ and the two edges leaving $\mathbf{v}$, one gets that (b) holds by applying Theorem (4.3).

$\Leftarrow$) Conversely, if P_1 and P_2 are complete intersections, let G_1 and G_2 be the two binary trees given by Theorem (4.3), denote by $\mathbf{v}_1$ and $\mathbf{v}_2$ their roots, and consider the binary tree G obtained by adding a vertex $\mathbf{v}$ and two edges leaving $\mathbf{v}$, one entering $\mathbf{v}_1$, the other entering $\mathbf{v}_2$. By (a), the vertex $\mathbf{v}$ of G (which is its root) satisfies the relation in Theorem (4.3), and any other non-terminal vertex of G satisfies it for being a non-terminal vertex of either G_1 or G_2, and hence P is a complete intersection. □

Remark (4.5). Given $d_1,\dots,d_n$ such that P is a complete intersection, a binary tree G labeled by $[\![1,n]\!]$ such that the arithmetical conditions of Theorem (4.3) are satisfied encodes the following information:

(i) The generators $\{g_1,\dots,g_{n-1}\}$ of P and their degrees $D_1,\dots,D_{n-1}$ can be obtained as shown in the proofs of Proposition (4.2) and Theorem (4.3).

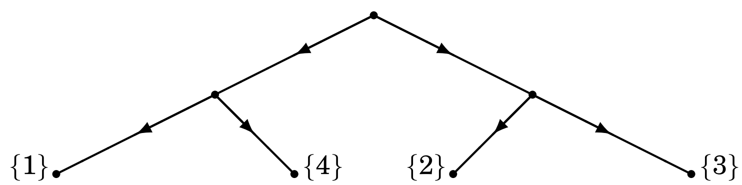
(ii) The Frobenius number $g(S)$ of the numerical semigroup $S = \mathbb{N}\underline{d}$, that is the largest integer not in S, can be expressed entirely in terms of $\{d_1,\dots,d_n\}$ when $\gcd(d_1,\dots,d_n) = 1$.

This last assertion is a consequence of the following. Recall that the quasi-homogeneous Hilbert series of R/P is $H_P(z) = \frac{f(z)}{(1-z^{d_1})\cdots(1-z^{d_n})}$ for some polynomial $f \in \mathbb{Z}[z]$. When $\gcd(d_1,\dots,d_n) = 1$, using that $R/P \simeq k[\Gamma]$, one can easily check that $H_P(z) = \frac{h(z)}{1-z}$ for some polynomial $h \in \mathbb{Z}[z]$ of degree $g(S)+1$. If P is a complete intersection, it is well-known that $f(z) = (1-z^{D_1})\cdots(1-z^{D_{n-1}})$ where $D_1,\dots,D_{n-1}$ are the degrees of the minimal quasi-homogeneous generators of P, and hence $g(S) = D_1 + \cdots + D_{n-1} - (d_1 + \cdots + d_n)$. Denoting by

$\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ the set of non terminal vertices of G and using (i), one gets the following formula:

$$g(S) = \left(\sum_{i=1}^{n-1} \frac{\gcd(d_j,\ j \in \ell_1[\mathbf{v}_i]) \gcd(d_j,\ j \in \ell_2[\mathbf{v}_i])}{\gcd(d_j,\ j \in \ell_1[\mathbf{v}_i] \cup \ell_2[\mathbf{v}_i])} \right) - \left(\sum_{i=1}^{n} d_i \right).$$

Example (4.6). Let k be an arbitrary field, and consider $d_1 = 16$, $d_2 = 27$, $d_3 = 45$ and $d_4 = 56$. The corresponding toric ideal $P \subset k[x_1, x_2, x_3, x_4]$ is a complete intersection because using the following binary tree labeled by $[\![1, 4]\!]$,



the arithmetical conditions in Theorem (4.3) are satisfied:

$$112 = \frac{(16)(56)}{\gcd(16, 56)} \in 16\mathbb{N} \cap 56\mathbb{N}: \qquad 2(56) \overset{(1)}{=} 7(16)$$

$$135 = \frac{(27)(45)}{\gcd(27, 45)} \in 27\mathbb{N} \cap 45\mathbb{N}: \qquad 3(45) \overset{(2)}{=} 5(27)$$

$$72 = \frac{\gcd(16, 56)\gcd(27, 45)}{\gcd(16, 27, 45, 56)} \in \{16, 56\}\mathbb{N} \cap \{27, 45\}\mathbb{N}:$$

$$1(16) + 1(56) \overset{(3)}{=} 1(27) + 1(45).$$

Moreover, the equalities (1), (2) and (3) provide, by Remark (4.5) (i), a set of minimal generators of P:

$$g_1 = x_4^2 - x_1^7, \quad g_2 = x_3^3 - x_2^5, \quad g_3 = x_1x_4 - x_2x_3\,.$$

Finally, by Remark (4.5) (ii), the Frobenius number of the numerical semigroup $S = \mathbb{N}\{16, 27, 45, 56\}$ is

$$g(S) = 112 + 135 + 72 - (16 + 27 + 45 + 56) = 175.$$

Remark (4.7). Toric ideals of affine monomial curves that are complete intersections were originally studied by Herzog in his paper [9]. In [9], Proposition 2.1, he considers the special situation where, after reindexing the d_i's if necessary, one has that

$$\text{(4.1)} \qquad \frac{\gcd(d_1, \ldots, d_i)\, d_{i+1}}{\gcd(d_1, \ldots, d_{i+1})} \in \mathbb{N}\{d_1, \ldots, d_i\},\ \forall i \in \{1, \ldots, n-1\},$$

and he wonders in the next remark if this property charaterizes the complete intersection case. The answer to this question is negative, this was first observed by K. Watanabe in [14], Remark 1, p. 105. In terms of binary trees, the situation in (4.1) corresponds to the case where $\#(\ell_2[\mathbf{v}]) = 1$ for each non-terminal vertex $\mathbf{v}$ of the binary tree involved in Theorem (4.3). Noting that in Theorem (4.3), one only needs to consider binary trees satisfying that $\#(\ell_1[\mathbf{v}]) \geq \#(\ell_2[\mathbf{v}])$ for any non-terminal vertex $\mathbf{v}$, it easily follows that, when

$n = 3$, P is a complete intersection if and only (4.1) holds after a suitable reindexing of the d_i's. This does not occur when $n \geq 4$. When $n = 4$, one has two possible binary trees satisfying that $\#(\ell_1[\mathbf{v}]) \geq \#(\ell_2[\mathbf{v}])$ for any non-terminal vertex $\mathbf{v}$, and one can check that in Example (4.6), there is no way of indexing the d_i's so that (4.1) hold. Indeed, for $n \geq 1$, the number τ_n of binary trees with n terminal vertices and satisfying that $\#(\ell_1[\mathbf{v}]) \geq \#(\ell_2[\mathbf{v}])$ for any non-terminal vertex $\mathbf{v}$, is given by the following inductive formula:

$$\tau_1 = \tau_2 = 1 \text{ and, for all } n \geq 3\,,\ \tau_n = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \tau_j \tau_{n-j}\,.$$

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ISABEL BERMEJO
DEPARTAMENTO DE MATEMÁTICA FUNDAMENTAL
FACULTAD DE MATEMÁTICAS
UNIVERSIDAD DE LA LAGUNA
38200-LA LAGUNA, TENERIFE
SPAIN
ibermejo@ull.es

PHILIPPE GIMENEZ
DEPARTAMENTO DE ÁLGEBRA, GEOMETRÍA Y TOPOLOGÍA
FACULTAD DE CIENCIAS
UNIVERSIDAD DE VALLADOLID
47005-VALLADOLID
SPAIN
pgimenez@agt.uva.es

ENRIQUE REYES AND RAFAEL H. VILLARREAL
DEPARTAMENTO DE MATEMÁTICAS
CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL IPN
APARTADO POSTAL 14–740
07000 MÉXICO, D.F.
vila@esfm.ipn.mx

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CLASSICAL VECTOR BUNDLES AND REPRESENTATIONS OF QUIVERS

PIOTR DOWBOR AND HAGEN MELTZER

ABSTRACT. We describe the Horrocks-Mumford bundle, null-correlation bundles and Tango bundles in terms of graded modules over the exterior algebra via the Bernstein-Gelfand-Gelfand correspondence. Furthermore we show that properties such as indecomposability and stability of these bundles can be proven purely algebraically.

Introduction

It is well known that vector bundles and coherent sheaves over certain projective algebraic varieties can be described in terms of linear algebra. In their fundamental papers [3] and [2] (see also [8]) Bernstein-Gelfand-Gelfand and Beilinson gave descriptions of the derived category of coherent sheaves on a projective space $\mathbb{P}^n$ over the field of complex numbers in terms of modules over finite dimensional algebras.

In this paper we will give algebraic descriptions of the Horrocks-Mumford bundle, null-correlation bundles and Tango bundles in terms of graded modules over the exterior algebra. These bundles are known to be indecomposable vector bundles of small rank and were investigated by rather geometrical methods (see [18] [13] [4] [20]). Here we study explicitly the corresponding modules of these bundles via the Bernstein-Gelfand-Gelfand correspondence. Furthermore, we show that indecomposability and stability can be shown by replacing advanced geometrical techniques by an investigation of the corresponding modules.

1. Basic facts and notations

The main aim of this section is to recall the results by Beilinson [2] and Bernstein-Gelfand-Gelfand [3], and to clarify the relationship between them expressed in terms of a result by Happel [10], as explained in [6].

(1.1) We briefly recall that Beilinson's result can be described using tilting theory as follows. Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be the direct sum of sheaves of twisted differential forms $\mathcal{F}_1 = \bigoplus_{0\le j\le n} \Omega^j(j)$ (resp. of twisted structure sheaves $\mathcal{F}_2 = \bigoplus_{0\le j\le n} \mathcal{O}(j)$) on the projective space $\mathbb{P}(V)$, where V is $(n+1)$-dimensional vector space. Denote by $E_i = \operatorname{End}(\mathcal{F}_i)$, $i = 1, 2$, the endomorphism rings. It is

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well known [2] that E_1 and E_2 can be in a natural way regarded as triangular matrix rings

$$A_1 = \begin{pmatrix} \Lambda^0(V) & \Lambda^1(V) & \dots & \Lambda^n(V) \\ 0 & \Lambda^0(V) & \dots & \Lambda^{n-1}(V) \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \Lambda^0(V) \end{pmatrix}, \quad A_2 = \begin{pmatrix} S^0(V^*) & 0 & \dots & 0 \\ S^1(V^*) & S^0(V^*) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ S^n(V^*) & S^{n-1}(V^*) & \dots & S^0(V^*) \end{pmatrix}$$

where $\Lambda(V) = \bigoplus_{0\le j\le n} \Lambda^j(V)$ and $S(V^*) = \bigoplus_{0\le j\le n} S^j(V^*)$ are the exterior algebra of the space V and symmetric algebras of the dual space V^*, respectively, endowed with the natural grading. Then, for $i = 1, 2$, the indecomposable direct summands of $\mathcal{F}_i$ generate the derived category $\mathcal{D}^b(\operatorname{coh}\mathbb{P}^n)$ of the category of coherent sheaves $\operatorname{coh}\mathbb{P}^n$. Moreover, we have $\operatorname{Ext}^s(\mathcal{F}_i, \mathcal{F}_i) = 0$ for $s > 0$ (see [2]). A coherent sheaf satisfying these conditions is called nowadays a tilting sheaf (compare [1]). It follows that the derived functor

$$\mathbb{L}G_i : \mathcal{D}^b(\operatorname{mod} A_i) \to \mathcal{D}^b(\operatorname{coh}\mathbb{P}^n)$$

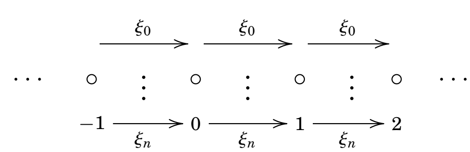
of the functor $G_i = - \otimes_{A_i} \mathcal{F}_i : \operatorname{mod} A_i \to \operatorname{coh}\mathbb{P}^n$ (under the identification $\operatorname{mod} E_i^{\mathrm{op}} \cong \operatorname{mod} A_i$, induced by the above isomorphisms), is an equivalence of triangulated categories where $\operatorname{mod} A$ denotes the category of finite dimensional left A-modules for any algebra A.

An alternative description of the derived category of coherent sheaves on projective spaces was given by Bernstein-Gelfand-Gelfand [3] (see also [8]). They proved that the functor $\Phi : \operatorname{mod}_{\mathbb{Z}}(\Lambda) \to \mathcal{D}^b(\operatorname{coh}\mathbb{P}^n)$, associating the complex

$$\Phi(M) : \quad \dots \to M_j \otimes \mathcal{O}(j) \to M_{j+1} \otimes \mathcal{O}(j+1) \to \dots$$

with each $\mathbb{Z}$-graded module $M = \bigoplus_{j\in\mathbb{Z}} M_j$ over the exterior algebra $\Lambda = \bigoplus_{0\le j\le n} \Lambda^j(V)$ induces an equivalence $\underline{\operatorname{mod}}_{\mathbb{Z}}(\Lambda) \cong \mathcal{D}^b(\operatorname{coh}\mathbb{P}^n)$, where $\underline{\operatorname{mod}}_{\mathbb{Z}}(\Lambda)$ denotes a factor category of the category $\operatorname{mod}_{\mathbb{Z}}(\Lambda)$ of all left $\mathbb{Z}$-graded Λ-modules by the ideal generated by projectives. This construction has been generalized in [17] and [16] and [19].

The two rather different descriptions of Bernstein-Gelfand-Gelfand and Beilinson are related by a result of Happel which states that, for any finite dimensional k-algebra A of finite global dimension, there is an equivalence $H : \mathcal{D}^b(\operatorname{mod} A) \to \underline{\operatorname{mod}}\hat{A}$ of triangulated categories where $\hat{A}$ denotes the repetitive algebra of A ($\hat{A}$ is selfinjective infinite dimensional) and $\underline{\operatorname{mod}}\hat{A}$ the stable category of $\hat{A}$-modules, that is, the category of finite dimensional $\hat{A}$-modules modulo the projectives [10]. (Recall that $\operatorname{mod} A$ can be regarded as a full subcategory of $\operatorname{mod}\hat{A}$ via the canonical "extension by zeros" embedding into a "zero component"). It is well known that $\operatorname{mod}\hat{A}_1 \cong \operatorname{mod}_{\mathbb{Z}}(\Lambda)$ (see [6]). Consequently, if k is the field $\mathbb{C}$ of complex numbers, $\hat{A}_1$-modules can be identified with $\mathbb{C}$-representations of the bound quiver (Q, I), where Q is the infinite quiver



and $I = \langle \xi_i^2, \xi_i\xi_j + \xi_j\xi_i \rangle$ (note that analogously A_1-modules can be regarded as $\mathbb{C}$-representations of the bound quiver which is a restriction of (Q, I) to the set of vertices $\{0, \dots n\}$).

Further on we will consider the tilting sheaf $\mathcal{F}_1 = \oplus_{i=0}^{n} \Omega^i(i)$. In this way we obtain the following triangle of equivalences

$$\begin{array}{ccccc} & & \mathcal{D}^b(\operatorname{mod} A) & & \\ & \swarrow^{H} & & \searrow^{\mathbb{L}G} & \\ \underline{\operatorname{mod}}\hat{A} & & \xrightarrow{\quad\Phi\quad} & & \mathcal{D}^b(\operatorname{coh} \mathbb{P}^n) \end{array}$$

where $A = A_1$ and $G = G_1$ (we keep this notation until the end of the paper). The diagram above does not commute, however in [6] a correction automorphism in each vertex, that makes the diagram commutative, is given. More precisely, define autoequivalences $\mathcal{S} : \mathcal{D}^b(\operatorname{coh} \mathbb{P}^n) \longrightarrow \mathcal{D}^b(\operatorname{coh} \mathbb{P}^n)$, $\mathcal{S}(-) = (-) \otimes \mathcal{O}(n+1)[-n]$, $\mathbb{R}\mathcal{N} : \mathcal{D}^b(\operatorname{mod} A) \longrightarrow \mathcal{D}^b(\operatorname{mod} A)$, where $\mathcal{N} = \operatorname{Hom}_A(-, A) \circ \operatorname{Hom}_k(-, k)$ is the inverse of the Nakayama functor, and $L(-) : \underline{\operatorname{mod}}\hat{A} \longrightarrow \underline{\operatorname{mod}}\hat{A}$, $L(-) = T(-)[n+1]$ where $(-)[1]$ denotes the translation functor in $\mathcal{D}^b(\operatorname{coh} \mathbb{P}^n)$ (respectively, shift of grading by 1 in $\underline{\operatorname{mod}}\hat{A}$) and for an $\hat{A}$-module M, $T(M)$ is given by an exact sequence $0 \longrightarrow M \longrightarrow E(M) \longrightarrow T(M) \longrightarrow 0$ in $\operatorname{mod}\hat{A}$ with $E(M)$ being the fixed (minimal) injective envelope of M (note that $T(M)$ is indecomposable if M is indecomposable nonprojective).

THEOREM (1.2) ([6]). (*i*) *The functors* $\Phi \circ H$, $\mathcal{S} \circ \mathbb{L}(G)$ *and* $\mathbb{L}(G) \circ \mathbb{R}\mathcal{N}$ *are isomorphic.*
(*ii*) *The functors* $\Phi \circ L \circ H$ *and* $\mathbb{L}(G)$ *are isomorphic.* □

(1.3) We work over the field of complex numbers. We will use the following notation. Let V be an $(n+1)$-dimensional $\mathbb{C}$-vector space with a fixed basis $v_0, v_1, \dots, v_n$, $\mathbb{P}^n = \mathbb{P}(V)$ the projective space and $\operatorname{coh} \mathbb{P}^n$ the category of coherent sheaves on $\mathbb{P}^n$. We will consider the category $\mathcal{D}^b(\operatorname{coh} \mathbb{P}^n)$ of bounded complexes of coherent sheaves of $\mathbb{P}^n$; for details concerning derived categories we refer to [9], [11] and [21]. Moreover, let $\mathcal{T}$ be the tangent bundle of $\mathbb{P}^n$ and $\Omega^s = \Lambda^s(\mathcal{T}^*)$ the s-th exterior power of the cotangent bundle.

For a vertex j of the ordinary quiver of A or $\hat{A}$ we denote by P_j, (resp. S_j, I_j), the standard indecomposable projective (resp. simple, indecomposable injective = dual of the standard right projective) left A- or $\hat{A}$-module respectively, where A- or an $\hat{A}$-module is always given as a representation of the corresponding bound quiver. The corresponding indecomposables over A and $\hat{A}$ obviously differ (via the canonical embedding $\operatorname{mod} A \subset \operatorname{mod}\hat{A}$), but the notation above will never lead to a confusion since we always precisely explain the context we work in.

For an $\hat{A}$-module $\mathbb{V}$ the rank is defined by $\operatorname{rk}(\mathbb{V}) = \sum (-1)^i \dim \mathbb{V}(i)$.

We will also need the following criterion saying which modules over $\hat{A}$ correspond to vector bundles on $\mathbb{P}^n$ [8]. Let $\mathbb{V} = ((\mathbb{V}(i)_{i\in\mathbb{Z}}, \mathbb{V}(\xi_k)_{\xi_k = \xi_k^i : i \to i+1, k=0,\dots n})$ be an $\hat{A}$-module. Recall, that $\mathbb{V}$ corresponds to the $\mathbb{Z}$ graded Λ-module $\bigoplus_i \mathbb{V}(i)$, where the multiplication by any vector $v = \sum_k c_k v_k \in V$ restricted to the i-th homogeneous component $\mathbb{V}(i)$ is given by the map $\mathbb{V}(v) = \mathbb{V}(v^i) = \sum_k c_k \mathbb{V}(\xi_k^i)$ for $i \in \mathbb{Z}$. For any v denote by $L_v(\mathbb{V})$ the complex of vector spaces

$$L_v : \qquad \dots \longrightarrow \mathbb{V}(-1) \xrightarrow{\mathbb{V}(v)} \mathbb{V}(0) \xrightarrow{\mathbb{V}(v)} \mathbb{V}(1) \longrightarrow \dots.$$

Definition (1.4). The module $\mathbb{V}$ is called *proper* if $H^j(L_v(\mathbb{V})) = 0$ for all $0 \neq v \in V$ and all $j \neq 0$.

THEOREM (1.5) ([8]). *$\mathbb{V}$ is a proper Λ-module if and only if $\Phi(\mathbb{V})$ is isomorphic in $\mathcal{D}^b(\operatorname{coh}\mathbb{P}^n)$ to a vector bundle.*

Example (1.6). a) The simple module S_m is not proper for $m \neq 0$. $\Phi(S_m)$ is the sheaf $\mathcal{O}(m)$, shifted to the place m. The module $T^m S_m$ is proper and $\Phi(T^m S_m) = \mathcal{O}(m)$.

b) The projective indecomposable P_j and injective indecomposable I_j corresponding to the vertex j are proper. They have the following form:

$$P_j = \left|\begin{array}{l|l} \Lambda^0 & j \\ \Lambda^1 & j{+}1 \\ \vdots & \vdots \\ \Lambda^n & j{+}n \\ \Lambda^{n+1} & j{+}n{+}1 \end{array}\right. \qquad I_j = \left|\begin{array}{l|l} (\Lambda^{n+1})^* & j{-}n{-}1 \\ (\Lambda^n)^* & j{-}n \\ \vdots & \vdots \\ (\Lambda^1)^* & j{-}1 \\ (\Lambda^0)^* & j \end{array}\right.$$

($P_j \cong I_{j+n+1}$). Note that in $\mathcal{D}^b(\operatorname{coh}\mathbb{P}^n)$ the objects $\Phi(P_j)$ and $\Phi(I_j)$ are zero.

c) Suppose that $n = 2$ and let M be the following module concentrated at the places -1 and 0:

$$\mathbb{C} \underset{\lambda}{\overset{\text{id}}{\rightrightarrows}} \mathbb{C}$$

where $\lambda \in \mathbb{C}$. Then M is indecomposable but not proper. Moreover, $\Phi(M)$ is quasi-isomorphic to the simple sheaf concentrated at λ.

2. The Horrocks-Mumford bundle

(2.1) In [13] Horrocks and Mumford discovered an indecomposable stable rank-2 vector bundle $\mathcal{F}_{HM}$ on $\mathbb{P}^4$ which is essentially the only one with this property. In fact, Decker and Schreyer proved in [4] that any stable rank-2 vector bundle on $\mathbb{P}^4$ is up to a line bundle twist and up to a pullback of an automorphism of $\mathbb{P}^4$ isomorphic to $\mathcal{F}_{HM}$. The Horrocks-Mumford bundle $\mathcal{F}_{HM}$ can be described as the cohomology of a monad, that is, three-term complex

$$0 \longrightarrow 5\Omega^4(4) \xrightarrow{b} 2\Omega^2(2) \xrightarrow{a} 5\mathcal{O} \longrightarrow 0$$

where $a = (a^{i,j})_{0\le i\le 4,\, 0\le j\le 1}$ is a 5×2 matrix with entries in $\Lambda^2(V)$ as follows $a^{i,0} = v_{i+2} \wedge v_{i+3}$ and $a^{i,1} = v_{i+1} \wedge v_{i+4}$ and $b = (aq)^t$, where $q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ [4]. Here $v_0, \dots, v_4$ denotes a basis of a 5 dimensional vector space V.

(2.2) Let $\mathbb{V}$ be the following $\hat{A}$-module of rank 2:

$$\underset{-1}{5\Lambda^0(V)} \xrightarrow[v_k \wedge b]{\quad\vdots\quad} \underset{0}{K} \xrightarrow[v_k]{\quad\vdots\quad} \underset{1}{2\Lambda^4(V)} \xrightarrow[v_k]{\quad\vdots\quad} \underset{2}{2\Lambda^5(V)}$$

where a and b are defined as in 2.1 and $K = \ker(2\Lambda^3 \xrightarrow{a} 5\Lambda^5)$. Note that $a^{i,j}$ are in the center of the algebra Λ.

It is easily checked that $\mathbb{V}$ is of rank 2 and is proper.

LEMMA (2.3). *There is no non-zero map $T^l S_l \longrightarrow \mathbb{V}$, for $l \geq 0$, in $\underline{\text{mod}}\,\hat{A}$.*

Proof. Let $f : T^l S_l \longrightarrow \mathbb{V}$ be an $\hat{A}$-homomorphism. Consider first the case $l = 0$. Then f is given by a nonzero linear map $f_0 : \mathbb{C} \longrightarrow K$ such that $\text{im} f_0 \subset \ker(v_0 \wedge -) \cap \cdots \cap \ker(v_4 \wedge -) = 0$, and consequently, $f = 0$.

To show the remaining cases we use information on the $\hat{A}$-modules $T^l S_l$, $l > 0$, one can derive from the injective resolutions of simple modules in mod $\hat{A}$, induced by the Koszul complex (see [5]). Recall that these resolutions have the form

$$0 \to S_l \to s_0 I_l \to s_1 I_{l-1} \to \ldots \to s_{l-1} I_1 \to \ldots$$

where $s_i = \dim_{\mathbb{C}} S^i(V^*)$ for $i \in \mathbb{N}$. Consequently, each $\hat{A}$-module $T^l S_l$, $l > 0$, is isomorphic to a direct summand (with a projective-injective complement) of cokernel of the differential $s_{l-2} I_2 \to s_{l-1} I_1$; and we have $T^l S_l = \text{Im}\,\pi_l$ for some $\hat{A}$-epimorphism $\pi_l : s_{l-1} I_1 \to T^l S_l$. Fix now $l > 0$. Then, due to projectivity of I_1, the composition $f\pi_l$ factors through the minimal projective cover $P(\mathbb{V})$ of the $\hat{A}$-module $\mathbb{V}$. Since $I_1 \cong P_{-4}$, $P(\mathbb{V}) \cong \bigoplus_{i=-1}^{2} d_i P_i$ for some $d_{-1}, \ldots, d_2 \in \mathbb{N}$, and $\text{Hom}_{\hat{A}}(P_{-4}, P_j) = 0$ for all $j \geq -3$, we infer $f = 0$ and the proof is complete. □

THEOREM (2.4). *$\Phi(\mathbb{V})$ is the Horrocks-Mumford bundle.*

Proof. We have to determine $\Phi(\mathbb{V}) = \mathbb{L}G \circ \mathbb{R}\mathbb{N} \circ H^{-1}(\mathbb{V})$. In the first step we show that $H^{-1}(\mathbb{V}) = X^\bullet$, equivalently, that $\mathbb{V} = H(X^\bullet) = T^{-1}(H(X^\bullet[1]))$, where $X^\bullet = (5I_4' \xrightarrow{b\cdot} 2I_2' \xrightarrow{a\cdot} 5I_0')$ is the complex concentrated between -1 and 1 and I_0', I_2' and I_4' are the left indecomposable injective A-modules as in the figure below (they differ from the standard indecomposable injective modules I_0, I_2 and I_4 defined in (1.3); note that entries of the matrices a and b act centrally on Λ since they belong to $\Lambda^2(V)$). For this purpose we compute $\mathbb{V}' = H(X^\bullet[1])$ using the following two pushout diagrams,

$$\begin{array}{ccc} 5I_4' & \longrightarrow & 2I_2' \\ \downarrow & & \downarrow \\ E_{\hat{A}}(5I_4') & \longrightarrow & C^{-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} C^{-1} & \longrightarrow & 5I_0' \\ \downarrow & & \downarrow \\ E_{\hat{A}}(C^{-1}) & \longrightarrow & C^0 = \mathbb{V} \end{array}$$

The calculation is given in the next diagram, where $Q = 2\Lambda^3/5\Lambda^5$ and $2\Lambda^2 \to Q$ is the embedding given by the matrix a.

$$\begin{array}{ccccc}
5I_4' = \left|\begin{array}{c|l} 5\Lambda^1 & 0\\ 5\Lambda^2 & 1\\ 5\Lambda^3 & 2\\ 5\Lambda^4 & 3\\ 5\Lambda^5 & 4\end{array}\right. & \xrightarrow{\;b\;} & 2I_2' = \left|\begin{array}{c|l} 2\Lambda^3 & 0\\ 2\Lambda^4 & 1\\ 2\Lambda^5 & 2\\ & 3\\ & 4\end{array}\right. & & \\
\downarrow & & \downarrow & & \\
E_{\tilde{A}}(5I_4') = \left|\begin{array}{c|l} 5\Lambda^0 & -1\\ 5\Lambda^1 & 0\\ 5\Lambda^2 & 1\\ 5\Lambda^3 & 2\\ 5\Lambda^4 & 3\\ 5\Lambda^5 & 4\end{array}\right. & \longrightarrow & C^{-1} = \left|\begin{array}{c|l} 5\Lambda^0 & -1\\ 2\Lambda^3 & 0\\ 2\Lambda^4 & 1\\ 2\Lambda^5 & 2\\ & 3\\ & 4\end{array}\right. & \xrightarrow{\;a\cdot\;} & 5I_0' = \left|\begin{array}{c|l} & -1\\ 5\Lambda^0 & 0\\ & 1\\ & 2\\ & 3\\ & 4\end{array}\right. \\
& & \downarrow & & \downarrow \\
& & E_{\tilde{A}}(C^{-1}) = \left|\begin{array}{c|l} 2\Lambda^0 & -3\\ 2\Lambda^1 & -2\\ 2\Lambda^2 & -1\\ 2\Lambda^3 & 0\\ 2\Lambda^4 & 1\\ 2\Lambda^5 & 2\end{array}\right. & \longrightarrow & \mathbb{V}' = \left|\begin{array}{c|l} 2\Lambda^0 & -3\\ 2\Lambda^1 & -2\\ Q & -1\\ 5\Lambda^5 & 0\\ & 1\\ & 2\end{array}\right.
\end{array}$$

Next by use of the exact sequence below we compute $T^{-1}(\mathbb{V}')$,

$$0 \longrightarrow T^{-1}(\mathbb{V}') \longrightarrow P(\mathbb{V}') \longrightarrow \mathbb{V}' \longrightarrow 0$$

$$\left|\begin{array}{c|l} & -3\\ & -2\\ 5\Lambda^0 & -1\\ K & 0\\ 2\Lambda^4 & 1\\ 2\Lambda^5 & 2\end{array}\right. \longrightarrow \left|\begin{array}{c|l} 2\Lambda^0 & -3\\ 2\Lambda^1 & -2\\ 2\Lambda^2 & -1\\ 2\Lambda^3 & 0\\ 2\Lambda^4 & 1\\ 2\Lambda^5 & 2\end{array}\right. \longrightarrow \left|\begin{array}{c|l} 2\Lambda^0 & -3\\ 2\Lambda^1 & -2\\ Q & -1\\ 5\Lambda^5 & 0\\ & 1\\ & 2\end{array}\right.$$

and the equality $H(X^\bullet) = \mathbb{V}$ is shown.

Now, we have

$$\begin{aligned}
\Phi(\mathbb{V}) &= \mathbb{L}G \circ \mathcal{N} \circ H^{-1}(\mathbb{V})) \\
&\cong \mathbb{L}G \circ \mathbb{R}\mathcal{N}(5I_4' \xrightarrow{b\cdot} 2I_2' \xrightarrow{a\cdot} 5I_0') \\
&\cong \mathbb{L}G \circ \mathcal{N}(5I_4 \xrightarrow{\nu(b)} 2I_2 \xrightarrow{\nu(a)} 5I_0) \\
&\cong \mathbb{L}G(5P_4 \xrightarrow{b} 2P_2 \xrightarrow{a} 5P_0) \\
&= (5\Omega^4(4) \xrightarrow{b} 2\Omega^2(2) \xrightarrow{a} 5\mathcal{O})
\end{aligned}$$

where $\nu = \mathrm{Hom}_k(-,k) \circ \mathrm{Hom}_A(-,A)$ is the Nakayama functor. Note that the algebra isomorphism $\sigma : \Lambda \to \Lambda$ induced under the standard identifications $\mathrm{End}_\Lambda({}_\Lambda\Lambda) \cong \Lambda^{\mathrm{op}}$, $\mathrm{End}_\Lambda((\Lambda_\Lambda)^*) \cong \Lambda^{\mathrm{op}}$ by the isomorphisms Θ' (see [6] 1.5 for the definition) is given for an arbitrary n by the mapping $\lambda \mapsto (-1)^{ni}\lambda$, $\lambda \in \Lambda^i(V)$,

$i \in \mathbb{N}$. Therefore we have the isomorphism $X^\bullet \cong (5I_4 \xrightarrow{\nu(b)} 2I_2 \xrightarrow{\nu(a)} 5I_0)$ of complexes. The remaining complex isomorphisms follow from the canonical isomorphisms $\mathrm{Hom}(\Omega^i(i), \Omega^j(j)) \cong \Lambda(V)^{i-j}$ defined by contraction [2] and the precise formula for the algebra isomorphisms $A^{\mathrm{op}} \cong \mathrm{End}(\bigoplus_{0\le j\le n} \Omega^j(j)) \cong \mathrm{End}(\bigoplus_{0\le j\le n} P_j)$. Thus we get a complex for $\Phi(\mathbb{V})$ which coincides with the monad description given in [4] (see 2.1). □

(2.5) Recall that a vector bundle $\mathcal{E}$ is called stable if for any coherent subsheaf $0 \neq \mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$ we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$, where μ denotes the slope.

The Horrocks-Mumford bundle is known to be stable [4]. We can give a simple proof by using Lemma 2.3.

COROLLARY (2.6). *The Horrocks-Mumford bundle is stable, in particular indecomposable.*

Proof. Since the Horrocks-Mumford bundle $\mathcal{F}_{HM}$ has rank 2 and slope $-\frac{1}{2}$, it is enough to show that there is no non-zero morphism $\mathcal{O}(l) \longrightarrow E$ with $l \geq 0$. Assume to the contrary that there is a non-zero morphism $g : \mathcal{O}(l) \longrightarrow E$. Since Φ is an equivalence, g is induced by a non-zero map $T^l S_l \longrightarrow \mathbb{V}$ in $\underline{\mathrm{mod}}\,\hat{A}$, contradicting Lemma 2.3. The last assertion follows from the well known fact that each stable bundle is indecomposable. □

Observe that $\mathbb{V}$ has no non-zero projective-injective direct summand so it is also indecomposable.

3. Null-correlation bundles

(3.1) For any odd $n > 1$ there is an indecomposable vector bundle $\mathcal{M}$ of rank $n-1$ on $\mathbb{P}^n$ defined as the kernel of a homomorphism

$$\mathcal{T}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}(1),$$

called the null-correlation bundle [18, 1.4.2].

In this section we define $\hat{A}$-modules, for each n, which in the odd case give an algebraic description of null-correlation bundles, in fact describe their duals. Further we investigate the properties of these modules and prove the indecomposability of the corresponding bundles by studying the endomorphism rings of the modules.

(3.2) Let $n \geq 1$ an integer and V a $n+1$-dimensional vector space with basis $v_0, \ldots, v_n$. Consider the $\hat{A}$-module $\mathbb{V}(\lambda)$ of rank $n-1$

$$\underset{-1}{\Lambda^0(V)} \;\underset{v_k\wedge\lambda}{\overset{\longrightarrow}{\vdots}}\; \underset{0}{\Lambda^n(V)} \;\underset{v_k}{\overset{\longrightarrow}{\vdots}}\; \underset{1}{\Lambda^{n+1}(V)}$$

where λ is an element of $\Lambda^{n-1}(V)$.

PROPOSITION (3.3). *Let $\lambda, \lambda' \in \Lambda^{n-1}(V)$. Then $\mathbb{V}(\lambda) \cong \mathbb{V}(\lambda')$ if and only if $\lambda' = k\lambda$ for some $k \neq 0$.*

Proof. Assume that $f : \mathbb{V}(\lambda) \longrightarrow \mathbb{V}(\lambda')$ is an isomorphism in $\underline{\mathrm{mod}}\,\hat{A}$. Then f is given by scalar multiplications $t : \Lambda^0(V) \longrightarrow \Lambda^0(V)$, $s : \Lambda^{n+1}(V) \longrightarrow \Lambda^{n+1}(V)$ with $s, t \neq 0$ and a $n \times n$ matrix B defining (in a standard basis of $\Lambda^n(V)$) a linear map $\Lambda^n(V) \longrightarrow \Lambda^n(V)$ such that the maps $v_k \wedge B$, $s \circ v_k : \Lambda^n(V) \longrightarrow \Lambda^{n+1}(V)$ and $B \circ (v_k \wedge \lambda)$, $(v_k \wedge \lambda') \circ t : \Lambda^0(V) \longrightarrow \Lambda^n(V)$ are respectively equal for every $k = 0, \dots, n$. Since $v_k : \Lambda^n(V) \longrightarrow \Lambda^{n+1}(V)$ is, up to sign, the k-th projection, the conditions $v_k \wedge B = s \circ v_k$ imply that $B = s \cdot \mathrm{id}$. Furthermore, if $\lambda = \sum_{i<j} c_{i,j}\, v_0 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n$ and $\lambda' = \sum_{i<j} c'_{i,j}\, v_0 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n$ we conclude from $s \circ (v_k \wedge \lambda) = (v_k \wedge \lambda') \circ t$ that $t \cdot c'_{i,j} = s \cdot c_{i,j}$ for all i, j. Therefore λ and λ' are proportional.

Now, assume that $\lambda' = k\lambda$, $k \neq 0$. Then $k^{-1} : \Lambda^0(V) \longrightarrow \Lambda^0(V)$, $\mathrm{id} : \Lambda^n(V) \longrightarrow \Lambda^n(V)$, $\mathrm{id} : \Lambda^{n+1}(V) \longrightarrow \Lambda^{n+1}(V)$ obviously defines an isomorphism $\mathbb{V}(\lambda) \longrightarrow \mathbb{V}(\lambda')$. □

PROPOSITION (3.4). $\underline{\mathrm{End}}_{\hat{A}}(\mathbb{V}(\lambda)) = \mathbb{C}$.

Proof. An endomorphism of $\mathbb{V}(\lambda)$ in $\mathrm{mod}\,\hat{A}$ is given by linear maps $t \colon \Lambda^0(V) \longrightarrow \Lambda^0(V)$, $B : \Lambda^n(V) \longrightarrow \Lambda^n(V)$, $s : \Lambda^{n+1}(V) \longrightarrow \Lambda^{n+1}(V)$ such that $B \circ (v_k \wedge \lambda) = (v_k \wedge \lambda) \circ t$ and $v_k \circ B = s \circ v_k$ for all $k = 0, \dots, n$. Similarly as in the proof of Proposition 3.3 one shows that $B = s \cdot \mathrm{id}$ and $s = t$, therefore $\mathrm{End}_{\hat{A}}(\mathbb{V}(\lambda)) = \mathbb{C}$, so $\underline{\mathrm{End}}_{\hat{A}}(\mathbb{V}(\lambda)) = \mathbb{C}$. □

(3.5) If $n = 1$ then $\mathbb{V}(\lambda)$ is given by a scalar $\lambda \in \mathbb{C}$. In this situation $\mathbb{V}(\lambda)$ is proper iff $\lambda \neq 0$ and $\Phi(\mathbb{V})$ is zero in $\mathcal{D}^b(\mathrm{coh}\,\mathbb{P}^n)$. In fact $\Phi(\mathbb{V})$ is the Auslander-Reiten sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \Lambda^1(V) \otimes \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

If $n = 2$ then $\mathbb{V}(\lambda)$ is given by an element $\lambda \in V$. Since $\lambda \wedge \lambda = 0$ it follows that $\mathbb{V}(\lambda)$ is never proper in this case.

We will now study the question of properness for arbitrary n. Let $\mathbb{V}(\lambda)$ be given by the element $\lambda = \sum_{i<j} c_{i,j}\, v_0 \wedge \dots \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n$. We define the following antisymmetric matrix

$$A(\lambda) = \begin{pmatrix} 0 & c_{0,1} & -c_{0,2} & \dots & \pm c_{0,n-2} & \mp c_{0,n-1} & \pm c_{0,n} \\ -c_{0,1} & 0 & +c_{1,2} & \dots & \mp c_{1,n-2} & \pm c_{1,n-1} & \mp c_{1,n} \\ c_{0,2} & -c_{1,2} & 0 & \dots & \pm c_{2,n-2} & \mp c_{2,n-1} & \pm c_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mp c_{0,n-2} & \pm c_{1,n-2} & \mp c_{2,n-2} & \dots & 0 & \pm c_{n-2,n-1} & \mp c_{n-2,n} \\ \pm c_{1,n-1} & \mp c_{1,n-1} & \pm c_{2,n-1} & \dots & \mp c_{n-2,n-1} & 0 & \pm c_{n-1,n} \\ \mp c_{0,n} & \pm c_{1.n} & \mp c_{2,n} & \dots & \pm c_{n-2,n} & \mp c_{n-1,n} & 0 \end{pmatrix}.$$

PROPOSITION (3.6). *The module* $\mathbb{V}(\lambda)$ *is proper if and only if* $\det(A(\lambda)) \neq 0$.

Proof. Since for any non-zero $v \in V$ the map $v : \Lambda^n(V) \longrightarrow \Lambda^{n+1}(V)$ is surjective we have that $\mathbb{V}(\lambda)$ is proper if and only if the map $v \wedge \lambda : \Lambda^0(V) \longrightarrow \Lambda^n(V)$ is injective, equivalently non-zero, for every non-zero vector $v \in V$. We set $\lambda = \sum_{i<j} c_{i,j}\, v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n$ and write $v = \sum_{i=0}^n a_i v_i$ as a linear combination the basis $v_0, v_1, \dots, v_n$ and fix the basis $(-1)^i v_0 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_n$, $i = 0, 1, \dots, n$, of $\Lambda^n(V)$. This leads to a homogeneous system of linear equations with unknowns $a_0, \dots, a_n$ having the matrix $A(\lambda)$ as coefficient

matrix. There is a non-trivial solution of this system if and only if $\det(A(\lambda)) = 0$. Consequently $\mathbb{V}(\lambda)$ is proper if and only if $\det(A(\lambda)) \neq 0$. □

COROLLARY (3.7). (i) *If n is even then there are no proper modules of the form $\mathbb{V}(\lambda)$.*

(ii) *If n is odd then there exists $\lambda \in \Lambda^{n-1}(V)$ such that $\mathbb{V}(\lambda)$ is a proper module.*

Proof. (i) For even n each antisymmetric $(n+1) \times (n+1)$ matrix has determinant zero.

(ii) In case n is odd, choosing all coefficients $c_{i,i+1} = 1$ and all $c_{i,j} = 0$ for $j > i+1$ the matrix $A(\lambda)$ has determinant 1. □

LEMMA (3.8). *Assume that $n = 3$. Then there is no non-zero map $T^l S_l \longrightarrow \mathbb{V}$, for $l \geq 0$, in $\underline{\operatorname{mod}} \hat{A}$.*

The proof is similar as that of Lemma 2.3. □

(3.9) For any module $\mathbb{V} = \mathbb{V}(\lambda)$, as defined in 3.2, we have an indecomposable complex $\Phi(\mathbb{V})$ of rank $n-1$ in $\mathcal{D}^b(\operatorname{coh} \mathbb{P}^n)$, it is of the form

$$0 \longrightarrow \Lambda^0(V) \otimes \mathcal{O}(-1) \longrightarrow \Lambda^n(V) \otimes \mathcal{O} \longrightarrow \Lambda^{n+1}(V) \otimes \mathcal{O}(1) \longrightarrow 0.$$

For proper modules we obtain the following result.

THEOREM (3.10). *Let n be odd and $\mathbb{V} = \mathbb{V}(\lambda)$ be proper. Then $\Phi(\mathbb{V})$ is a dual of a null-correlation bundle.*

Proof. We have $\Phi(\mathbb{V}) = \mathbb{L}G \circ \mathbb{R}\mathcal{N} \circ H^{-1}(\mathbb{V})$. The calculation of $H^{-1}(\mathbb{V})$ is similar as in the case of the Horrocks-Mumford bundle (see the proof of Theorem (2.4)). The result is presented in the following diagram

$$\begin{array}{ccc}
I'_n = \left| \begin{array}{l} \Lambda^1 \\ \Lambda^2 \\ \Lambda^3 \\ \vdots \\ \Lambda^n \\ \Lambda^{n+1} \end{array} \right| \begin{array}{l} 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{array} & \xrightarrow{\lambda\cdot} & I'_1 = \left| \begin{array}{l} \Lambda^n \\ \Lambda^{n+1} \\ \\ \vdots \\ \\ \\ \end{array} \right| \begin{array}{l} 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{array} \\
\downarrow & & \downarrow \\
E_{\tilde{A}}(I'_n) = \left| \begin{array}{l} \Lambda^0 \\ \Lambda^1 \\ \Lambda^2 \\ \vdots \\ \Lambda^n \\ \Lambda^{n+1} \end{array} \right| \begin{array}{l} -1 \\ 0 \\ 1 \\ \vdots \\ n-1 \\ n \end{array} & \longrightarrow & \mathbb{V} = \left| \begin{array}{l} \Lambda^0 \\ \Lambda^n \\ \Lambda^{n+1} \\ \vdots \\ \\ \\ \end{array} \right| \begin{array}{l} -1 \\ 0 \\ 1 \\ \vdots \\ n-1 \\ n \end{array}
\end{array}$$

(λ acts centrally since $n-1$ is even). Now, by the analogous arguments as in (2.4),

$$\begin{aligned}\Phi(\mathbb{V})) &= \mathbb{L}G \circ \mathbb{R}\mathcal{N} \circ H^{-1}(\mathbb{V}) \\ &\cong \mathbb{L}G \circ \mathcal{N}(I'_n \xrightarrow{\lambda\cdot} I'_1) \\ &\cong \mathbb{L}G \circ \mathcal{N}(I_n \xrightarrow{\nu(\lambda)} I_1) \\ &\cong \mathbb{L}G(P_n \xrightarrow{\lambda} P_1) \\ &\cong (\Omega^n(n) \xrightarrow{\lambda} \Omega^1(1)).\end{aligned}$$

Since $\mathbb{V} = \mathbb{V}(\lambda)$ is a proper module, the complex $\Phi(\mathbb{V}(\lambda)) \cong \Omega^n(n) \xrightarrow{\lambda} \Omega^1(1)$, $\lambda \in \mathrm{Hom}(\Omega^n(n), \Omega^1(1)) \cong \Lambda^{n-1}(V)$, is quasi-isomorphic to the bundle $\mathcal{L}(\lambda)$ which fits in an exact sequence

$$0 \longrightarrow \Omega^n(n) \xrightarrow{\lambda} \Omega^1(1) \longrightarrow \mathcal{L}(\lambda) \longrightarrow 0.$$

Now the result follows from [18], 1.4.2, by dualization. □

Observe that if $\mathcal{M}$ is a null-correlation bundle then its dual is always of the form $\mathcal{M}(\lambda)^* \cong \Phi(\mathbb{V}(\lambda))$, where $\lambda \in \Lambda^{n-1}(V)$ is such that $\mathbb{V}(\lambda)$ is proper.

PROPOSITION (3.11). *For $n = 3$ each null-correlation bundle is stable, in particular indecomposable.*

Proof. Since a vector bundle is stable if and only if its dual is stable, it suffices to show that each bundle $\Phi(\mathbb{V})$ is stable, where $\mathbb{V}$ is as in 3.2. Because $\mu(\Phi(\mathbb{V})) = 0$ and $\mathrm{rk}(\Phi(\mathbb{V})) = 2$ it is enough to know that there is no non-zero map $\mathcal{O}(l) \to \Phi(\mathbb{V})$, for $l \geq 0$. This follows from Lemma 3.8 as in Corollary 2.6. □

4. Tango bundles

(4.1) For arbitrary n Tango gave a description of indecomposable vector bundles $\mathcal{E}$ on $\mathbb{P}^n$ of rank $n-1$ [20] (see also [7], [15], [14], [18] I.4.2). These bundles are given by exact sequences of the form

$$(*) \qquad 0 \longrightarrow m\mathcal{O} \xrightarrow{w} (\Lambda^2\mathcal{T})(-2) \longrightarrow \mathcal{E}(1) \longrightarrow 0.$$

where $m = \binom{n-1}{2}$ and w represents m general sections of $(\Lambda^2\mathcal{T})(-2)$, thus $w = (w_1, \ldots, w_m)$ with $w_i \in \Lambda^2(V)$.

Dualizing the exact sequence $(*)$ we obtain

$$0 \longrightarrow \mathcal{E}(1)^\star \longrightarrow \Omega^2(2) \xrightarrow{g} m\mathcal{O} \longrightarrow 0$$

where $g = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$.

In this section we define certain $\hat{A}$-modules and show that the proper ones among them correspond to duals of Tango bundles. We study the question of properness of these modules purely algebraically and reprove the stability of Tango bundles in dimension 4 by investigation of the corresponding modules.

(4.2) For any $m \in \mathbb{N}$ and $w = (w_1, \ldots, w_m) \in \Lambda^2(V)^m$ consider the following $\hat{A}$-module $\mathbb{V} = \mathbb{V}(w)$:

$$\underset{-n+2}{\Lambda^0(V)} \; \underset{v_k}{\overset{\longrightarrow}{\vdots}} \; \underset{-n+3}{\Lambda^1(V)} \; \underset{v_k}{\overset{\longrightarrow}{\vdots}} \; \cdots \; \underset{v_k}{\overset{\longrightarrow}{\vdots}} \; \underset{-1}{\Lambda^{n-3}(V)} \; \underset{v_k}{\overset{\longrightarrow}{\vdots}} \; \underset{0}{\Lambda^{n-2}(V)} \; \underset{v_k \wedge g}{\overset{\longrightarrow}{\vdots}} \; \underset{1}{m\Lambda^{n+1}(V)}$$

(g is as above). In the case $m = \binom{n-1}{2}$, $\mathbb{V}$ is of rank $n-1$.

PROPOSITION (4.3). *The $\hat{A}$-module $\mathbb{V}=\mathbb{V}(w)$ is proper if and only if $w_1, \ldots, w_m$ are linearly independent and no non-trivial linear combination $c_1w_1 + \cdots + c_mw_m$ is simple. (Recall that $u \in \Lambda^2(V)$ is called simple if $u = v \wedge v'$ for some $v, v' \in \mathbb{V}$, equivalently $u \wedge u = 0$ as an element of $\Lambda^4(V)$).*

Proof. We have that $\mathbb{V}$ is proper if and only if the map

$$v \wedge g = v \wedge \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} : \Lambda^{n-2}(V) \longrightarrow m\Lambda^{n+1}(V)$$

is surjective for any $v \neq 0 \in V$. Note that the necessary condition for the surjectivity of the map above is the inequality $m \leq \binom{n}{2}$. For a given vector $0 \neq v \in V$ fix a basis $v'_0 = v, v'_1, \ldots, v'_n$ of V. The map $v \wedge g$ is surjective if and only if the rank of the matrix

$$C = \begin{pmatrix} +c_1^{1,2} & \ldots & +c_m^{1,2} \\ -c_1^{1,3} & \ldots & -c_m^{1,3,} \\ \vdots & & \vdots \\ +c_1^{n-1,n} & \ldots & +c_m^{n-1,n} \end{pmatrix}$$

is equal to m, where $w_k = \sum_{0 \leq i < j \leq n} c_k^{i,j} v'_i \wedge v'_j$, $k = 1, \ldots, m$. (Note that the matrix of $v \wedge g$ in the standard bases has the form $\binom{0}{C}^{\mathrm{tr}}$). This immediately implies that if $\mathbb{V}$ is proper then $w_1, \ldots, w_m$ are linearly independent.

Suppose that $\mathbb{V}$ is proper. Let $u = c_1w_1 + \cdots + c_mw_m$ be a non-zero linear combination which is simple, say $u = v' \wedge v''$ for some $v', v'' \in V$. Then we can choose $c'_1, \ldots, c'_n \in \mathbb{C}$ such that $c_1c'_1 + \cdots + c_mc'_m \neq 0$. Since the map $v' \wedge g$ is surjective by assumption there is $t \in \Lambda^{n-2}(V)$ such that

$$v' \wedge \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} \wedge t = \begin{pmatrix} c'_1 \\ \vdots \\ c'_m \end{pmatrix}$$

(we identify canonically $\Lambda^{n+1}(V)$ with $\mathbb{C}$). It follows that $v' \wedge (c_1w_1 + \cdots + c_mw_m) \wedge t = c_1c'_1 + \cdots + c_mc'_m \neq 0$. On the other hand $v' \wedge (c_1w_1 + \cdots + c_mw_m) \wedge t = v' \wedge (v' \wedge v'') \wedge t = 0$, a contradiction.

Suppose now that $\mathbb{V}$ is not proper, i.e., $v \wedge g$ is not surjective. Then without loss of generality we can assume $c_m^{i,j} = \sum_{k=1}^{m-1} \alpha_k c_k^{i,j}$, $1 \leq i < j \leq n$, for some $\alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$. It follows that $\sum_{k=1}^{m-1} \alpha_k w_k - w_m$ is either zero or of the form $v'_0 \wedge \bar{v}$, where $\bar{v} = \sum_{k=1}^{m-1} \alpha_k (\sum_{j=1}^{n} c_k^{0,j} v'_j) - \sum_{j=1}^{n} c_m^{0,j} v'_j$, hence is simple. □

From now on we assume $m = \binom{n-1}{2}$.

THEOREM (4.4). *If $\mathbb{V} = \mathbb{V}(w)$ is proper, then $\Phi(\mathbb{V})$ is the dual of a Tango bundle.*

Proof. Set $\mathbb{V}' = \mathbb{V}[1]$. We have $\Phi(\mathbb{V}') = \mathbb{L}G \circ \mathbb{R}\mathcal{N} \circ H^{-1}(\mathbb{V}')$. The complex $H^{-1}(\mathbb{V}')$ is isomorphic to the complex $I_2' \xrightarrow{g} mI_0'$, concentrated between -1 and 0; the calculation is presented in the following diagram:

$$
\begin{array}{ccc}
I_2' = \left|\begin{array}{l|l} \Lambda^{n-1} & 0 \\ \Lambda^{n} & 1 \\ \Lambda^{n+1} & 2 \\ & \vdots \\ & n-1 \\ & n \\ & n+1 \end{array}\right. & \xrightarrow{\;g\cdot\;} & mI_0' = \left|\begin{array}{l|l} m\Lambda^{n+1} & 0 \\ & 1 \\ & 2 \\ & \vdots \\ & n-1 \\ & n \\ & n+1 \end{array}\right. \\
\downarrow & & \downarrow \\
E_{\tilde{A}}(I_2') = \left|\begin{array}{l|l} \Lambda^{0} & -n+1 \\ \Lambda^{1} & -n+2 \\ \Lambda^{2} & -n+3 \\ \vdots & \vdots \\ \Lambda^{n-1} & 0 \\ \Lambda^{n} & 1 \\ \Lambda^{n+1} & 2 \end{array}\right. & \longrightarrow & \mathbb{V}' = \left|\begin{array}{l|l} \Lambda^{0} & -n+1 \\ \Lambda^{1} & -n+2 \\ \Lambda^{2} & -n+3 \\ \vdots & \vdots \\ m\Lambda^{n+1} & 0 \\ & 1 \\ & 2 \end{array}\right.
\end{array}
$$

(see (2.4) for the notation). Hence

$$
\begin{aligned}
\Phi(\mathbb{V}') &= \mathbb{L}G \circ \mathbb{R}\mathcal{N} \circ H^{-1}(\mathbb{V}) \\
&\cong \mathbb{L}G \circ \mathcal{N}(I_2' \xrightarrow{g} mI_0') \\
&\cong \mathbb{L}G \circ \mathcal{N}(I_2 \xrightarrow{\nu(g)} mI_0) \\
&\cong \mathbb{L}G\,(P_2 \xrightarrow{g} mP_0) \\
&\cong (\Omega^2(2) \xrightarrow{g} m\mathcal{O}).
\end{aligned}
$$

Here the complex $\Omega^2(2) \xrightarrow{g} m\mathcal{O}$ is concentrated between -1 and 0, therefore $\Phi(\mathbb{V}') \cong E^*(-1)[1]$.

Now, $\mathbb{V} = \mathbb{V}'[-1] \cong \mathbb{V}' \otimes S_1$ where S_1 is the $\hat{A}$-module corresponding to the vertex 1. Because Φ commutes with the tensor product we conclude $\Phi(\mathbb{V}) \cong \Phi(\mathbb{V}' \otimes S_1) \cong \Phi(\mathbb{V}') \otimes \Phi(S_1) \cong \mathcal{E}^\star(-1)[1] \otimes \mathcal{O}(1)[-1] \cong \mathcal{E}^\star$ and the proof is complete. □

(4.5) By the preceding result the dual of Tango bundle is quasi-isomorphic to a complex

$$0 \longrightarrow \Lambda(V)^0 \otimes \mathcal{O}(-n+2) \longrightarrow \ldots \longrightarrow \Lambda(V)^{n-2} \otimes \mathcal{O} \longrightarrow m\Lambda(V)^{n+1} \otimes \mathcal{O}(1) \longrightarrow 0.$$

For $n = 3$ a Tango bundle is a null-correlation bundle. If $n = 4$ then a Tango bundle is given by 3 elements $w_1, w_2, w_3 \in \Lambda^2(V)$. Investigating the corresponding module we show that the dual of a Tango bundle on $\mathbb{P}^4$ is stable. For a general proof using geometrical arguments we refer to [14] 8.6.

LEMMA (4.6). *Let $\mathbb{V}$ be a proper $\hat{A}$-module such that $\Phi(\mathbb{V}) = \mathcal{E}^*$, $\deg(\mathcal{E}^*) = 0$ and $\mathrm{rk}(\mathcal{E}^*) = 3$. Assume that there is no non-zero morphism $T^l S_l \longrightarrow \mathbb{V}$ for $l \geq 0$ and no non-zero morphism $\mathbb{V} \longrightarrow T^l S_l$ for $l \leq 0$ in $\underline{\mathrm{mod}}\,\hat{A}$. Then $\mathcal{E}$ is stable.*

Proof. We prove that $\mathcal{E}^*$ is stable. By [18], II, 1.2.2, it is sufficient to show that for any coherent subsheaf $\mathcal{F}$ of $\mathcal{E}^*$ such that $\mathcal{E}^*/\mathcal{F}$ is torsion free and $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E}^*)$ we have $\mu(\mathcal{F}) < \mu(\mathcal{E}^*) = 0$. Suppose contrary that there is a subsheaf $\mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E}^*)$ and $\mu(\mathcal{F}) \geq \mu(\mathcal{E}^*)$ such that $\mathcal{E}^*/\mathcal{F}$ is torsion free.

Assume first that $\mathrm{rk}(\mathcal{F}) = 1$. Consider an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Since $\mathcal{E}^*$ is locally free and $\mathcal{Q}$ is torsion free, $\mathcal{F}$ is reflexive by [18] II, 1.1.16 and 1.1.12. Now $\mathrm{rk}(\mathcal{F}) = 1$ implies that $\mathcal{F}$ is locally free, hence of the form $\mathcal{O}(l)$ for some $l \in \mathbb{Z}$ with $l \geq 0$. But then the morphism $\mathcal{O}(l) \longrightarrow \mathcal{E}^*$ is the image of some non-zero morphism $T^l S_l \longrightarrow \mathbb{V}$ in $\underline{\mathrm{mod}}\,\hat{A}$, contrary to our assumption.

Assume now that $\mathrm{rk}(\mathcal{F}) = 2$. Consider again an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Since $\mathcal{Q}$ is torsion free, $\mathcal{Q}^{\star\star}$ is reflexive and of rank one, hence $\mathcal{Q}^{\star\star} = \mathcal{O}(l)$ for some $l \in \mathbb{Z}$. Now $\mu(\mathcal{F}) \geq$ implies that $l \leq 0$. Then the morphism $\mathcal{E}^* \longrightarrow \mathcal{Q} \hookrightarrow \mathcal{Q}^{\star\star} = \mathcal{O}(l)$ is the image of some non-zero morphism $\mathbb{V} \longrightarrow T^l S_l$ in $\underline{\mathrm{mod}}\,\hat{A}$, which again is a contradiction to our assumption. Therefore $\mathcal{E}^*$ is stable, and consequently, $\mathcal{E}$ is so. □

THEOREM (4.7). *A Tango bundle on $\mathbb{P}^4$ is stable, in particular indecomposable.*

Proof. Observe first that $\mathrm{rk}(\mathcal{E}^*) = 3$ and $\deg(\mathcal{E}^*) = 0$. According to the lemma above we have to show that there is no non-zero map in $\underline{\mathrm{mod}}\,\hat{A}$, $T^l S_l \longrightarrow \mathbb{V}$, for $l \geq 0$ and no non-zero map $\mathbb{V} \longrightarrow T^l S_l$, for $l \leq 0$.

We prove first that there is no a non-zero map $S_0 \to \mathbb{V}$. For this we will show that the map $h = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} : \Lambda^2(V) \longrightarrow 3\Lambda^4(V)$ is injective. This is sufficient, because a non-zero morphism $f : S_0 \longrightarrow \mathbb{V}$, defined by a linear map $f_0 : \mathbb{C} \longrightarrow \Lambda^2(V)$, gives $w_r \wedge f_0(1) \neq 0$ for some r. Therefore $v_k \wedge w_r \wedge f_0(1) \neq 0$ for some k. We conclude that $0 = f_1 \circ S_0(\xi_k)(1) = \mathbb{V}(\xi_k) \circ f_0(1) = v_k \wedge \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \wedge f_0(1) \neq 0$ which gives a contradiction.

Suppose first that $h(\lambda) = 0$ for some non-zero simple element $\lambda \in \Lambda^2(V)$. Suppose that $\lambda = v'_0 \wedge v'_1$ for some basis $v'_0, v'_1, v'_2, v'_3, v'_4$. Denoting $w_r = \sum_{0 \leq i < j \leq 4} c_r^{i,j} v'_i \wedge v'_j$, $r = 1, 2, 3$, $w_r \wedge \lambda = 0$ implies $c_r^{i,j} = 0$ for $r = 1, 2, 3$ and $\{i, j\} \cap \{0, 1\} = \emptyset$. Hence $w_r = v'_0 \wedge p_r + v'_1 \wedge q_r$ where p_r belongs to the linear hull of v'_1, v'_2, v'_3, v'_4 and q_r belongs to the linear hull of v'_2, v'_3, v'_4. Now, if q_1, q_2, q_3 are linearly dependent, some non-trivial combination is of the form $\alpha w_1 + \beta w_2 + \gamma w_3 = v'_0 \wedge (\alpha p_1 + \beta p_2 + \gamma p_3)$, a contradiction to Proposition 4.3 and the assumption that $\mathbb{V}$ is proper. Thus $v'_0, v'_1, q_1, q_2, q_3$ is a basis of $\mathbb{V}$ and changing v_2', v_3', v_4' by q_1, q_2, q_3 we can assume that $w_r = c_r^{0,1}v_{0,1}' + c_r^{0,2}v_{0,2}' + c_r^{0,3}v_{0,3}' + c_r^{0,4}v_{0,4}' + v_{1,r+1}'$, $r = 1, 2, 3$, where $v_{i,j}' = v_i' \wedge v_j'$ for $0 \leq i < j \leq 4$. Then, for $v = v_1' - sv_0'$, $s \in \mathbb{C}$, the matrix of the map $v \wedge \left(\begin{smallmatrix} w_1 \\ w_2 \\ w_3 \end{smallmatrix}\right) : \Lambda^2(V) \longrightarrow 3\Lambda^5(V)$ in the basis $v_{i,j}'$ has only the following non-zero columns:

$$\begin{pmatrix} -c_1^{0,2} - s & c_1^{0,3} & -c_1^{0,4} \\ -c_2^{0,2} & c_2^{0,3} + s & -c_2^{0,4} \\ -c_3^{0,2} & c_3^{0,3} & -c_3^{0,4} - s \end{pmatrix}.$$

Obviously there is an $s \in \mathbb{C}$ such that $v \wedge \left(\begin{smallmatrix} w_1 \\ w_2 \\ w_3 \end{smallmatrix}\right)$ is not surjective, contrary to the assumption that $\mathbb{V}$ is proper.

Suppose now that $h(\lambda) = 0$ for some non-zero non-simple element $\lambda \in \Lambda^2(V)$. It is well known that $\lambda \in \Lambda^2(V)$ as a non-simple element has the form $\lambda = v_0' \wedge v_1' + v_2' \wedge v_3'$ for some basis $v_0', v_1', v_2', v_3', v_4'$ of V. Now $w_r \wedge \lambda = 0$ implies that $c_r^{0,1} = 0$, $c_r^{2,3} = 0$ and $c_r^{i,4} = 0$, $i = 1, \ldots 4$, for every $r = 1, 2, 3$ (we keep the previous notation). Then for $v = v_1'$ the matrix of the map $v \wedge \left(\begin{smallmatrix} w_1 \\ w_2 \\ w_3 \end{smallmatrix}\right) : \Lambda^2(V) \longrightarrow 3\Lambda^5(V)$ has only the following non-zero columns:

$$\begin{pmatrix} c_1^{0,3} & -c_1^{0,2} \\ c_2^{0,3} & -c_2^{0,2} \\ c_3^{0,3} & -c_3^{0,2} \end{pmatrix}.$$

Consequently, this map has rank 2, so is not surjective contrary to our assumption. Therefore there is no non-zero morphism $S_0 \longrightarrow \mathbb{V}$.

Suppose now that there is a non-zero morphism $f : \mathbb{V} \longrightarrow S_0$ in $\underline{\mathrm{mod}}\hat{A}$. It is given by a non-zero map $f_0 : \Lambda^2(V) \longrightarrow \mathbb{C}$. Then $f_0 \circ v_k \neq 0$ for some k. It follows that $0 = S_0(\xi_k) \circ f_{-1} = f_0 \circ \mathbb{V}(\xi_k) \neq 0$, a contradiction.

Now it remains to show that for any $l > 0$ there are no non-zero maps $T^l S_l \to \mathbb{V}$ and $\mathbb{V} \to T^{-l} S_{-l}$.

In the first case we apply exactly the same technique as in the proof of Lemma 2.2. In our current situation the minimal projective cover $P(\mathbb{V})$ of the $\hat{A}$-module $\mathbb{V}$ has the form $P(\mathbb{V}) \cong \bigoplus_{i=-n+2}^{1} d_i P_i$, $d_{-n+2}, \ldots, d_1 \in \mathbb{N}$, and $I_1 \cong P_{-n}$; moreover, $\mathrm{Hom}_{\hat{A}}(P_{-n}, P_j) = 0$ for all $j \geq -n+1$. Then repeating the arguments one concludes again that each $\hat{A}$-homomorphism $f : T^l S_l \to \mathbb{V}$ is a zero map, whenever $l > 0$.

To prove the second claim we use the fact that the simple $\hat{A}$-modules S_l, $l \in \mathbb{Z}$, admit projective resolutions of the form

$$\cdots \to s_{l-1} P_{-1} \to \cdots \to s_1 P_{-l+1} \to s_0 P_{-l} \to S_{-l} \to 0.$$

Note that they can be easily obtained from the resolutions in (2.3) by applying the standard duality and the isomorphism $\hat{A} \cong \hat{A}^{\mathrm{op}}$ induced by the mapping $i \mapsto -i$, $i \in \mathbb{Z}$. Now the dual arguments to those from (2.3) provide that for any $l > 0$, the $\hat{A}$-module $T^{-l} S_{-l}$ can be regarded as an $\hat{A}$-submodule of $s_{l-1} P_{-1}$. Denote by ι_l the embedding $T^{-l} S_{-l} \hookrightarrow s_{l-1} P_{-1}$. Then for $f : \mathbb{V} \to T^{-l} S_{-l}$, $l > 0$, the map $\iota_l f$ factors through the (minimal) injective envelope $I(\mathbb{V})$ of the $\hat{A}$-module $\mathbb{V}$. Since $P_{-1} \cong I_n$, $I(\mathbb{V}) \cong \bigoplus_{i=-n+2}^{1} d_i I_i$ for some $d_{-n+2}, \ldots, d_1 \in \mathbb{N}$, and $\mathrm{Hom}_{\hat{A}}(I_j, I_n) = 0$ for all $j < n$, we immediately infer $f = 0$ $(n > 1)$ and the proof is complete. □

PIOTR DOWBOR,
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE,
NICOLAUS COPERNICUS UNIVERSITY,
87-100 TORUŃ,
POLAND
E-mail: dowbor@mat.uni.torun.pl

HAGEN MELTZER,
INSTITUTE OF MATHEMATICS,
SZCZECIN UNIVERSITY,
70-451 SZCZECIN,
POLAND
E-mail: meltzer@wmf.univ.szczecin.pl

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LIFT CATEGORIES AND OBJECTS WITH SPACE OF SELFEXTENSIONS OF DIMENSION ONE

EFRÉN PÉREZ

ABSTRACT. We study a natural exact structure on lift categories and properties of the corresponding bimodule of extensions. These extensions behave nicely under the action of reduction functors. A lift category and a tensor product of algebras induce a new lift category, and its properties are studied. The existence is proved of a parametrization of the objects with space of selfextensions of k-dimension one, when k is algebraically closed.

1. Introduction

The bimodules over categories with structure of coalgebra (bocs) have been used to prove important theorems in the theory of representations of finite dimensional algebras over algebraically closed fields, like Drozd's Tame and Wild Theorem. The idea behind the use of bocses is the existence of reduction functors, which allows us to go from a representation of a bocs to a representation with smaller norm in another bocs. Despite the simplicity of this idea, bocses are not the most tractable of objects so they have produced very interesting results only for finite dimensional algebras over an algebraically closed field.

There have been several efforts in order to generalize the reduction functors, see [9], [10] and [5]. In the first two papers the concept of lift category is developed and it is used to deal with representations of Artin algebras of finite representation type and to handle generic modules.

Let us recall some definitions.

Definition (1.1). A lift pair (R, ξ) is given by a ring R and an exact sequence of R-bimodules

$$\xi\colon\ 0 \longrightarrow M \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } R \longrightarrow 0$$

Definition (1.2). Given a lift pair (R, ξ) we define the lift category $\xi\,(R)$ as follows: the objects are pairs (P, e) where P is a projective R-module and $e : P \to E \otimes_R P$ is an R-morphism such that the composition

$$P \quad \xrightarrow{\ e\ } \quad E \otimes_R P \xrightarrow{\pi\otimes 1} \quad R \otimes_R P \xrightarrow{\ \cong\ } \quad P$$

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is 1_P. A morphism $f:(P,e)\to(P',e')$ is an R-morphism $f:P\to P'$ such that the following diagram is commutative:

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ {}^{e}\downarrow & & \downarrow^{e'} \\ E\otimes_R P & \xrightarrow{1\otimes f} & E\otimes_R P' \end{array}$$

An object (P,e) in $\xi(R)$ is called *finite* if and only if P is a finitely generated R-module.

We can use this very general setting to study $\mathrm{mod}\,\Lambda$ for an Artin algebra Λ, because there is a natural lift pair associated, and the corresponding lift category is equivalent to $P^1(\Lambda)$ (7.2).

In this paper I follow the path traced in [6] and [7] in order to get all the strength of the ideas contained there.

In the second section we recall the main results about lift categories and reduction functors, stressing the advantages of using dual bases of projective modules. In the third section the exact structure for lift categories introduced in [7] provides the tool of extensions, which was used in [7] to study objects with trivial space of selfextensions, and inductiveness in [6].

An important fact is that reduction functors induce morphisms between the extensions and we see this in Section 4. A nice result is Lemma (4.3) where it is proved that the reduction functor associated to the anhilation of the radical, J, induces an isomorphism when $MJ=0=JM$; this is necessary if we want to study objects with non-trivial selfextensions. Theorem (4.6), proved in [7], gives a simple relation between the extensions of objects when we apply the reduction functors associated to an R-R-sub-bimodule N of M and a fixed finite object X.

In Section 5 there is introduced the construction of a lift pair from a given lift pair, when R is a k-algebra, and a k-algebra S using tensorization. Here, particularly in Propositions (5.4) and (5.6) we see that the behavior of an adequate subcategory of the new lift category is nearly componentwise, a property very useful and easy to handle. This and Theorem (4.6) allow us to prove the main result of this paper, Theorem (6.10); the existence of parametrizations for objects with space of selfextensions of k-dimension one when k is algebraically closed.

From (6.10) there follows Corollary (7.6) which proves, for a finite dimensional algebra over an algebraically closed field, that we can parametrize a family of modules, given by a fixed dimension and a space of selfextensions of dimension one in its minimal projective resolution, without any assumption on the representation type. Corollary (7.6) was first proved in [4] using bocses, but that paper still is unpublished.

If we add the hypothesis of tame representation type to Corollary (7.6) then it can be obtained in a similar way to Drozd's Theorem [8], or even generalized to finite dimensional algebras over perfect fields of non-wild type using bocses [13]. Tameness provides a lot of simplification in using bocses, so it is hard to prove Corollary (7.6) with this tool in the wild case. Then lift categories are a good alternative, because they deal easily with objects with space of selfextensions of k-dimension one, for k algebraically closed; we can get a glance of this claim in Lemma (6.4).

2. Lift categories and reduction functors

By R we denote a ring, by R-Mod the left R-modules and by R-mod the finitely generated left R-modules. If M belongs to R-Mod, we denote $M^* = \mathrm{Hom}_R(M,R)$. For $M,N \in R$-Mod we often write ${}_R(M,N)$ instead of $\mathrm{Hom}_R(M,N)$. For X,Y in $\xi(R)$ we often write ${}_{\xi(R)}(X,Y)$ instead of $\mathrm{Hom}_{\xi(R)}(X,Y)$.

Remark (2.1). For a projective R-module P fix a dual base $(\lambda_l, x_l)_{l\in I}$, where $\lambda_l \in P^*$ and $x_l \in P$. Fix $\omega \in E$ such that $\pi(\omega) = 1_R$; we can define the R-morphism $\nu_\omega : P \to E \otimes_R P$ by $\nu_\omega(p) = \sum \lambda_l(p)\,\omega \otimes x_l$, then $(P,\nu_\omega) \in \xi(R)$. Now, the exact sequence of groups

$$0 \quad\to\quad {}_R(P, M\otimes_R P) \quad\xrightarrow{(i\otimes 1)_*}\quad {}_R(P, E\otimes_R P) \quad\xrightarrow{(\pi\otimes 1)_*}\quad {}_R(P,P) \quad\to\quad 0$$

shows that for an object (P,e) in $\xi(R)$, there is an unique R-morphism $u : P \to M\otimes_R P$ such that $e = \nu_\omega + (i\otimes 1)u$.

Let us observe that for any $r \in R$, $r\omega = \omega r + i\Delta(r)$, where $\Delta : R \to M$ is a derivation; i.e., Δ is additive and $\Delta(rs) = \Delta(r)s + r\Delta(s)$.

Now, if (P_1,e_1) and (P_2,e_2) are objects in $\xi(R)$, and we have the dual bases $(\lambda_l,x_l)_{l\in I}$ and $(\mu_j,y_j)_{j\in J}$, of P_1 and P_2 respectively, then there are R-morphisms $\nu_\omega^k : P_k \to E\otimes_R P_k$ and $u_k : P_k \to M\otimes_R P_k$ such that $e_k = \nu_\omega^k + (i\otimes 1)u_k$. Moreover, for an R-morphism $f : P_1 \to P_2$ we have

$$e_2 f - (1\otimes f)e_1 = \sum_{l,j} \lambda_l(\_)\, i\Delta(\mu_j(f(x_l))) \otimes y_j + (i\otimes 1)u_2 f - (i\otimes f)u_1 \tag{2.2}$$

Definition (2.3). A *morphism of lift pairs* $\phi : (R,\xi) \to (R',\xi')$ is a pair $\phi = (\phi_0,\phi_1)$ such that $\phi_0 : R \to R'$ is a morphism of rings, $\phi_1 : E \to E'$ is a morphism of R-R-bimodules, where the structure of R-R-bimodule of E' is induced by ϕ_0, and such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\pi} & R \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\phi_0} \\ E' & \xrightarrow{\pi'} & R' \end{array}$$

Lemma (2.4). [6] *Every morphism of lift pairs ϕ: $(R,\xi) \to (R',\xi')$ induces a functor $F_\phi : \xi(R) \to \xi'(R')$. Given an object (P,e) in $\xi(R)$, we have $F_\phi(P,e) = (R'\otimes_R P, e')$, where e' is the composition*

$$\begin{array}{ccccc} R'\otimes_R P & \xrightarrow{1\otimes e} & R'\otimes_R E\otimes_R P & \xrightarrow{\cong} & R'\otimes_R E\otimes_R R\otimes_R P \\ e'\downarrow & & & & \downarrow{\scriptstyle 1\otimes\phi_1\otimes\phi_0\otimes 1} \\ E'\otimes_{R'} R'\otimes_R P & \xleftarrow{\cong} & R'\otimes_{R'} E'\otimes_{R'} R'\otimes_R P & \xleftarrow{\zeta\otimes 1} & R'\otimes_R E'\otimes_R R'\otimes_R P \end{array}$$

and $\zeta : R'\otimes_R E'\otimes_R R' \to R'\otimes_{R'} E'\otimes_{R'} R'$ is the canonical morphism, and the isomorphisms are induced by the corresponding multiplications. The functor F_ϕ maps each morphism $f : (P_1,e_1) \to (P_2,e_2)$ in $\xi(R)$ onto $F_\phi(f) = 1\otimes f : (R'\otimes_R P_1, e_1') \to (R'\otimes_R P_2, e_2')$.

Remark (2.5). Observe that composition above is equivalent to the composition

$$(\alpha_{E'}\otimes 1)(1\otimes\phi_1\otimes 1)(1\otimes e) : R'\otimes_R P \to E'\otimes_R P$$

where $\alpha_{E'} : R'\otimes_R E' \to E'$ is the action of R' on E'.

As is shown in [6], an isomorphism of lift pairs $\phi : (R, \xi) \to (R', \xi')$ induces an equivalence of categories $F_\phi : \xi(R) \to \xi'(R')$.

Now let (P, ν_ω) be the object of $\xi(R)$ determined by $\omega \in E$ and the dual base (λ_i, x_i) of P. Denote by $\alpha : R' \otimes_R R \to R'$ the canonical isomorphism. Observe $(\alpha(1 \otimes \lambda_i), 1 \otimes x_i)$ is a dual base of $R' \otimes_R P$. This dual base and the element $\phi_1(\omega)$ induce the R'-morphism $\nu_{\phi_1(\omega)}$, given by $\nu_{\phi_1(\omega)}(t \otimes p) = \sum_i \alpha(t \otimes \lambda_i(p)) \phi_1(\omega)(1 \otimes x_i) = \sum_i t\phi_1(\lambda_i(p)\omega) \otimes (1 \otimes x_i)$.

On the other hand, any morphism of lift pairs $\phi = (\phi_0, \phi_1) : (R, \xi) \to (R', \xi')$ induces a commutative diagram of R-R-bimodules

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{i} & E & \xrightarrow{\pi} & R & \to & 0 \\ & & \downarrow^{\phi_M} & & \downarrow^{\phi_1} & & \downarrow^{\phi_0} & & \\ 0 & \to & M' & \xrightarrow{i'} & E' & \xrightarrow{\pi'} & R' & \to & 0 \end{array}$$

A straightforward computation shows that, if $(P, e) \in \xi(R)$ where $e = \nu_\omega + (i \otimes 1)u$, then $F_\phi(P, e) = (P', e')$, where $e' = \nu_{\phi_1(\omega)} + (i' \otimes 1)u'$, u' is the composition

$$R' \otimes_R P \xrightarrow{1 \otimes u} R' \otimes_R M \otimes_R P \xrightarrow{1 \otimes \phi_M \otimes 1} R' \otimes_R M' \otimes_R P \xrightarrow{\alpha_{M'} \otimes 1} M' \otimes_R P$$

and $\alpha_{M'} : R' \otimes_R M' \to M'$ is the action of R' on M'.

Proposition (2.6). [9] *The forgetful functor $F_0 : \xi(R) \to R$-Proj is faithful and dense and reflects isomorphisms. Moreover, if $f : P \to P'$ is an isomorphism of projective R-modules, and (P, e) is an object in $\xi(R)$, then there is an object (P', e') in $\xi(R)$ such that f induces an isomorphism $(P, e) \to (P', e')$.*

Now we recall some results about reduction functors between lift categories.

Let (R, ξ) be a lift pair and I an ideal of R, then there is an associated lift pair

$$\xi_I : \quad 0 \to \frac{M}{M \cap (EI + IE)} \to \frac{E}{EI + IE} \to \frac{R}{I} \to 0$$

Moreover, the morphism of lift pairs

$$\begin{array}{ccc} E & \xrightarrow{\pi} & R \\ {}^{\pi_1}\downarrow & & \downarrow^{\pi_0} \\ \frac{E}{EI+IE} & \xrightarrow{\pi'} & \frac{R}{I} \end{array}$$

induces a functor $F_I : \xi(R) \to \xi_I(R/I)$.

We recall that an ideal I in a ring R is said to be *left T-nilpotent* if, given any sequence $(i_1, i_2, ...)$ of elements of I, the product $i_1 i_2 ... i_n$ is zero for some n. A ring is *left perfect* if and only if every left module has a projective cover. A useful characterization is that R is left perfect if and only if $R/\operatorname{rad} R$ is artinian and $\operatorname{rad} R$ is left T-nilpotent.

Lemma (2.7). [9] *Let I be a left T-nilpotent ideal of R. If M is a simple R-R-bimodule then $IM = MI = 0$.*

Theorem (2.8). [9] *Let (R, ξ) be a lift pair and I an ideal of R. If I is a left T-nilpotent ideal in R satisfying $IM = MI = 0$ then the functor $F_I : \xi(R) \to \xi_I(R/I)$ is a representation equivalence (full, dense and reflects isomorphisms) which preserves coproducts. Moreover $F_I(P, e)$ is finite if and only if (P, e) is finite.*

If (R,ξ) is a lift pair and N is an R-R-sub-bimodule of M, then there is an associated lift pair

$$\xi_N: \quad 0 \to \frac{M}{N} \to \frac{E}{N} \to R \to 0$$

a morphism of lift pairs

$$\begin{array}{ccc} E & \xrightarrow{\pi} & R \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle 1} \\ \frac{E}{N} & \xrightarrow{\pi'} & R \end{array}$$

and an induced functor $F_N : \xi(R) \to \xi_N(R)$.

Theorem (2.9). [9] *Let (R,ξ) be a lift pair and N an R-R-sub-bimodule of M. The functor $F_N : \xi(R) \to \xi_N(R)$ is faithful and dense, reflects isomorphisms and preserve coproducts. $F_N(P,e)$ is finite if and only if (P,e) is finite.*

Let now (R,ξ) be a lift pair, N an R-R-sub-bimodule of M, and $X = (P,\underline{e})$ a finite object in $\xi_N(R)$. We denote $R_X = \left(\mathrm{End}_{\xi_N(R)}(X)\right)^{op}$. P is naturally an R-R_X-bimodule, and we obtain a lift pair through the pullback diagram

$$\begin{array}{ccccccccc} \xi_X : 0 & \to & M_X & \to & E_X & \to & R_X & \to & 0 \\ & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \to & {}_R(P, N \otimes_R P) & \to & {}_R(P, E \otimes_R P) & \to & {}_R\left(P, \frac{E}{N} \otimes_R P\right) & \to & 0 \end{array}$$

where each space is considered as an R_X-R_X-bimodule and β is the bimodule map sending 1 to $\underline{e}$.

In this situation there is a functor $\tau_X : \xi_X(R_X) \to \xi(R)$, given in objects by $\tau_X(Q,g) = (P \otimes_{R_X} Q, h)$, where h is the composition

$$\begin{array}{ccc} P \otimes_{R_X} Q & \xrightarrow{1\otimes g} & P \otimes_{R_X} E_X \otimes_{R_X} Q \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1\otimes\alpha\otimes 1} \\ E \otimes_R P \otimes_{R_X} Q & \xleftarrow{\underline{v}\otimes 1} & P \otimes_{R_X} \mathrm{Hom}_R(P, E \otimes_R P) \otimes_{R_X} Q \end{array}$$

and $\underline{v}$ is the evaluation map. If $f : (Q_1, g_1) \to (Q_2, g_2)$ is a morphism in $\xi_X(R_X)$ then $\tau_X(f) = 1 \otimes f : P \otimes_{R_X} Q_1 \to P \otimes_{R_X} Q_2$ is a morphism in $\xi(R)$.

Theorem (2.10). [9] *Let (R,ξ) be a lift pair and N an R-R-sub-bimodule of M. The functor $\tau_X : \xi_X(R_X) \to \xi(R)$ is fully faithful. It induces an equivalence from $\xi_X(R_X)$ to the full subcategory of $\xi(R)$ on those objects whose image under F_N is isomorphic to a summand of a coproduct of copies of X. Moreover, $\tau_X(Q,g)$ is finite if and only if (Q,g) is finite. Additionally, if R is left perfect then R_X is left perfect.*

Remark (2.11). [9] A lift category $\xi(R)$ is an additive category, with arbitrary coproducts and split idempotents. If R is semiperfect, i.e., any finitely generated object has a projective cover, then the subcategory of finite objects constitutes a Krull-Schmidt category.

3. Exact structures

Definition (3.1). Let $\xi(R)$ be a lift category and $F_0 : \xi(R) \to R$-Proj the forgetful functor. We define the class ϵ of sequences $Y \xrightarrow{i} Z \xrightarrow{d} X$ in $\xi(R)$ such that the sequence $0 \to F_0(Y) \to F_0(Z) \to F_0(X) \to 0$ is exact. An element (i,d) of ϵ will be called a *conflation.*

Proposition (3.2). [7] *ϵ is an exact structure.*

Remark (3.3). An important fact about exact structures is that there is a commutative diagram of conflations

$$\begin{array}{ccccc} Y & \rightarrow & Z & \rightarrow & X \\ \parallel & & \downarrow^{\theta} & & \parallel \\ Y & \rightarrow & W & \rightarrow & X \end{array}$$

if and only if θ is an isomorphism, so we have the usual equivalence relation.

Definition (3.4). Let (R,ξ) be a lift pair, X and Y objects in $\xi(R)$. By $\mathrm{Ext}_{\xi}(X,Y)$ will be denoted the equivalence classes of conflations $Y \overset{i}{\rightarrow} Z \overset{d}{\rightarrow} X$.

Remark (3.5). Another important fact on exact structures is the existence of pullbacks and pushouts ([11]) thus $\mathrm{Ext}_{\xi}(X,Y)$ has the usual structure of $\mathrm{End}_{\xi}(Y) - \mathrm{End}_{\xi}(X)$-bimodule. In fact, more generally, there is a bifunctor $\mathrm{Ext}_{\xi}(\_,?) : \xi(R)^{op} \times \xi(R) \rightarrow \mathrm{Ab}$.

Proposition (3.6). [7] *Let $\xi(R)$ be a lift category and $X = (P_1, e_1)$ and $Y = (P_2, e_2)$ objects in $\xi(R)$.*

There is an exact sequence

$$\begin{array}{ccccc} 0 \rightarrow & \mathrm{Hom}_{\xi}(X,Y) & \overset{\sigma_{X,Y}}{\longrightarrow} & \mathrm{Hom}_{R}(P_1,P_2) & \overset{\delta_{X,Y}}{\longrightarrow} \\ & \mathrm{Hom}_{R}(P_1, M \otimes_R P_2) & \overset{\eta_{X,Y}}{\longrightarrow} & \mathrm{Ext}_{\xi}(X,Y) & \rightarrow 0 \end{array}$$

where $\sigma_{X,Y}$ is the canonical inclusion, $\delta_{X,Y}(f) = e_2 f - (1 \otimes f) e_1$ and $\eta_{X,Y}(h)$ is the class of the sequence $Y \overset{(1_{P_2},0)^t}{\longrightarrow} Z \overset{(0,1_{P_1})}{\longrightarrow} X$ where $Z = (P_2 \oplus P_1, e)$ and $e = \begin{pmatrix} e_2 & (i \otimes 1) h \\ 0 & e_1 \end{pmatrix}$.

Moreover, σ, δ and η are natural transformations in the variables X and Y.

4. Exact structures and reduction functors

In this section we study how conflations behave under reduction functors. Most of the results were proved in [7] but here we present statements slightly different, stressing the presevation of the bimodule structure. This new statements can be obtained from the original ones.

Remark (4.1). A functor $F : C \rightarrow D$ between exact categories is said to be exact if it sends conflations to conflations. By [11] it is known that an exact functor sends pullbacks to pullbacks and pushouts to pushouts, therefore an exact functor induces a morphism of bimodules between extensions.

It is immediate that the reduction functors are exact, so we will study the properties of the induced morphisms.

Lemma (4.2). *Let $\phi : (\xi, R) \rightarrow (\xi', R')$ be a morphism of lift pairs, $F_{\phi} : \xi(R) \rightarrow \xi'(R')$ the induced functor, $X = (P_1, e_1)$, and $Y = (P_2, e_2)$ objects in*

$\xi(R)$, *and* $F_\phi(X) = X'$, $F_\phi(Y) = Y'$. *There is a commutative diagram of exact sequences*

$$\begin{array}{cccccc}
0 \to & \mathrm{Hom}_\xi(X,Y) & \stackrel{\sigma_{X,Y}}{\longrightarrow} & \mathrm{Hom}_R(P_1,P_2) & \stackrel{\delta_{X,Y}}{\longrightarrow} \\
 & \downarrow^{F_\phi} & & \downarrow^{1\otimes -} & \\
0 \to & \mathrm{Hom}_{\xi'}(X',Y') & \stackrel{\sigma_{X',Y'}}{\longrightarrow} & \mathrm{Hom}_{R'}(R'\otimes_R P_1, R'\otimes_R P_2) & \stackrel{\delta_{X',Y'}}{\longrightarrow}
\end{array}$$

$$\begin{array}{cccc}
\mathrm{Hom}_R(P_1, M\otimes_R P_2) & \stackrel{\eta_{X,Y}}{\longrightarrow} & \mathrm{Ext}_\xi(X,Y) & \to 0 \\
\downarrow^{\theta} & & \downarrow^{\psi_\phi} & \\
\mathrm{Hom}_{R'}(R'\otimes_R P_1, M'\otimes_R P_2) & \stackrel{\eta_{X',Y'}}{\longrightarrow} & \mathrm{Ext}_{\xi'}(X',Y') & \to 0
\end{array}$$

where $\theta(u)$ *is the composition of Remark (2.5) and* ψ_ϕ *is the morphism of* $\mathrm{End}_{\xi(R)}(Y) - \mathrm{End}_{\xi(R)}(X)$*-bimodules between extensions induced by* F_ϕ.

Proof. It is easy to see the commutativity of the diagram. By Remark (2.5) the functor F_ϕ sends a sequence of the form $Y \stackrel{(1_{P_2},0)^t}{\longrightarrow} Z \stackrel{(0,1_{P_1})}{\longrightarrow} X$ where $Z = (P_2 \oplus P_1, e)$ and $e = \begin{pmatrix} e_2 & (i\otimes 1)h \\ 0 & e_1 \end{pmatrix}$, in the sequence

$Y' \stackrel{(1_{R'\otimes_R P_2},0)^t}{\longrightarrow} Z' \stackrel{(0,1_{R'\otimes P_1})}{\longrightarrow} X'$ where $Z = (R'\otimes_R P_2 \oplus R'\otimes_R P_1, e')$ and $e' = \begin{pmatrix} e_2' & (i'\otimes 1)\theta(h) \\ 0 & e_1' \end{pmatrix}$. It follows that ψ_ϕ makes the diagram commutative and by Proposition (3.6) it is a morphism of bimodules. □

In order to simplify notation, in the next lemma we denote $M^J = \frac{M}{JM} \cong \frac{R}{J}\otimes_R M$, by $\nu_P : P \to P^J$ the canonical epimorphism, and by $G_J : R\text{–Mod} \to \frac{R}{J}\text{–Mod}$ the functor given in objects by $G_J(M) = M^J$, and in morphisms by sending the R-morphism $f : M \to N$ in the induced $\frac{R}{J}$-morphism $f' : M^J \to N^J$.

The next lemma improves Theorem (4.1) of [7].

Lemma (4.3). *Let* (R,ξ) *be a lift pair,* J *the radical of* R, $m_J = M \cap (JE + EJ)$, $\pi_J : M \to M_J = M/m_J$ *the canonical epimorphism. Then, for any* $X = (P_1, e_1)$ *and* $Y = (P_2, e_2)$ *in* $\xi(R)$, *there is a commutative diagram,* $(F_J(X) = X', F_J(Y) = Y')$,

$$\begin{array}{cccccc}
0 \to & {}_{\xi(R)}(X,Y) & \stackrel{\sigma_{X,Y}}{\longrightarrow} & {}_R(P_1,P_2) & \stackrel{\delta_{X,Y}}{\longrightarrow} \\
 & \downarrow^{F_J} & & \downarrow^{G_J} & \\
0 \to & {}_{\xi_J(R)}(X',Y') & \stackrel{\sigma_{X',Y'}}{\longrightarrow} & {}_{\frac{R}{J}}\left(P_1^J, P_2^J\right) & \stackrel{\delta_{X',Y'}}{\longrightarrow}
\end{array}$$

$$\begin{array}{cccc}
{}_R(P_1, M\otimes_R P_2) & \stackrel{\eta_{X,Y}}{\longrightarrow} & \mathrm{Ext}_\xi(X,Y) & \to 0 \\
\downarrow^{G_J(\pi_J\otimes 1)_*} & & \downarrow^{\psi_J} & \\
{}_{\frac{R}{J}}\left(P_1^J, M_J\otimes_{\frac{R}{J}} P_2^J\right) & \stackrel{\eta_{X',Y'}}{\longrightarrow} & \mathrm{Ext}_{\xi_J}(X',Y') & \to 0
\end{array}$$

where ψ_J *is the epimorphism of* $\mathrm{End}_{\xi(R)}(Y) - \mathrm{End}_{\xi(R)}(X)$*-bimodules induced by* F_J.

If $MJ = 0 = JM$ *then* ψ_J *is an isomorphism.*

Proof. We recall that, if $S \to T$ is an epimorphism of rings, then the canonical functor $H : T - \mathrm{Mod} \to S - \mathrm{Mod}$ is a full embedding. So we have that $\mathrm{Hom}_{\frac{R}{J}}\left(P_1^J, M_J \otimes_{\frac{R}{J}} P_2^J\right) = \mathrm{Hom}_R\left(P_1^J, M_J \otimes_{\frac{R}{J}} P_2^J\right)$. Moreover, because

P_1 is projective, for any g in $\mathrm{Hom}_R\left(P_1^J, M_J \otimes_{\frac{R}{J}} P_2^J\right)$ there is an R-morphism $f : P_1 \to M \otimes_R P_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} P_1 & \xrightarrow{f} & M \otimes_R P_2 & & \\ \downarrow^{\nu_{P_1}} & & \downarrow^{\pi_J \otimes 1} & & \\ P_1^J & \xrightarrow{g} & M_J \otimes_R P_2 & \cong & M_J \otimes_{\frac{R}{J}} P_2^J \end{array}$$

Thus $G_J(\pi_J \otimes 1)_*$ is an epimorphism. In a similar way we prove that the second vertical morphism is an epimorphism.

The commutativity of the diagram and the existence of ψ_J follows by Lemma (4.2).

It is easy to see that the kernel of ψ_J is the image under $\eta_{X,Y}$ of the kernel of $G_J(\pi_J \otimes 1)_*$.

Now let us assume that $MJ = 0 = JM$. We fix $\omega \in E$ in the preimage of 1_R. Then $M \cap (JE + EJ) = \Delta(J) = m_J \subset M$, where $\Delta : R \to M$ is the derivation of Remark (2.1), given by $r\omega = \omega r + i\Delta(r)$. Observe that $\Delta : J \to m_J$ is a surjective R-R-morphism.

If $g_0 : P_1 \to M \otimes_R P_2$ is a morphism in the kernel of $G_J(\pi_J \otimes 1)_*$, then it factors through a morphism $g : P_1 \to m_J \otimes_R P_2$. Also there is a morphism $f_0 : P_1 \to J \otimes_R P_2$ such that $(\Delta \otimes 1) f_0 = g$. We define $f : P_1 \to P_2$ as the composition $P_1 \xrightarrow{f_0} J \otimes_R P_2 \cong JP_2 \hookrightarrow P_2$.

Now we fix dual bases (λ_l, x_l) and (μ_j, y_j) for P_1 and P_2 respectively. By formula (2.2),

$$\begin{aligned} (e_2 f - (1 \otimes f) e_1)(p) &= \textstyle\sum_{l,j} \lambda_l(p)\, i\Delta(\mu_j f(x_l)) \otimes y_j + (u_2 f - (i \otimes f) u_1)(p) \\ &= \textstyle\sum_{l,j} i\Delta(\lambda_l(p)\, \mu_j f(x_l)) \otimes y_j \\ &= (i \otimes 1) \textstyle\sum_j \Delta(\mu_j f(p)) \otimes y_j \\ &= (i \otimes 1)(\Delta \otimes 1) f_0(p) = (i \otimes 1) g(p) \,. \end{aligned}$$

We have $u_2 f = 0 = (i \otimes f) u_1$ because $MJ = 0 = JM$. It follows that $g \in \mathrm{Im}\, \delta_{X,Y}$, so ψ_J is injective. □

LEMMA (4.4). [7] *Let (R, ξ) be a lift pair, N an R-R-sub-bimodule of M and $\pi_N : M \to \frac{M}{N}$ the canonical epimorphism, and $F_N : \xi(R) \to \xi_N(R)$ the associated reduction functor. For any $X = (P_1, e_1)$ and $Y = (P_2, e_2)$ in $\xi(R)$, $F_N(X) = X'$ and $F_N(Y) = Y'$, we have a commutative diagram,*

$$\begin{array}{ccccccc} 0 \to & {}_{\xi(R)}(X,Y) & \xrightarrow{\sigma_{X,Y}} & {}_R(P_1,P_2) & \xrightarrow{\delta_{X,Y}} & {}_R(P_1, M \otimes_R P_2) \\ & \downarrow^{F_N} & & \| & & \downarrow^{(\pi_N \otimes 1)_*} \\ 0 \to & {}_{\xi_N(R)}(X',Y') & \xrightarrow{\sigma_{X',Y'}} & {}_R(P_1,P_2) & \xrightarrow{\delta_{X',Y'}} & {}_R\left(P_1, \frac{M}{N} \otimes_R P_2\right) \end{array}$$

$$\begin{array}{ccc} \xrightarrow{\eta_{X,Y}} & \mathrm{Ext}_{\xi(R)}(X,Y) & \to 0 \\ & \downarrow^{\psi_N} & \\ \xrightarrow{\eta_{X',Y'}} & \mathrm{Ext}_{\xi_N(R)}(X',Y') & \to 0 \end{array}$$

where ψ_N is the epimorphism of $\mathrm{End}_{\xi(R)}(Y) - \mathrm{End}_{\xi(R)}(X)$-bimodules induced by F_N.

For the next lemma let C be an R_X-module, A an R_X-module, B an R-R_X-bimodule and P an R-R_X-bimodule R-projective and finitely generated. The evaluation $\underline{v} : P \otimes_{R_X} \mathrm{Hom}_R(P,B) \to B$ induces a natural isomorphism

$$_{R_X}(C,_R(P,B) \otimes_{R_X} A) \quad \cong \quad _R(P \otimes_{R_X} C, B \otimes_{R_X} A)$$

Now let Q_1, Q_2 be R_X-projective modules and N an R-R-bimodule. The previous isomorphism induces the morphisms

$\phi_1 :_{R_X}(Q_1,Q_2) \cong_{R_X} (Q_1, 1_p \otimes_{R_X} Q_2) \to_R (P \otimes_{R_X} Q_1, P \otimes_{R_X} Q_2)$ and
$\phi_2 :_{R_X}(Q_1,_R(P, N \otimes_R P) \otimes_{R_X} Q_2) \overset{\cong}{\to}_R (P \otimes_{R_X} Q_1, N \otimes_R P \otimes_{R_X} Q_2)$.

Lemma (4.5). [7] *Let (R,ξ) be a lift pair, N an R-R-sub-bimodule of M, $i_N : N \to M$ the inclusion and $X = (P, e_N)$ a finite object in $\xi_N(R)$. Let $\tau_X : \xi_X(R_X) \to \xi(R)$ be the fully faithful functor of Theorem (2.9). Then there is a canonical induced functor $\tau_X : \xi_X(R_X) \to \xi(R)$. For any $Z = (Q_1, v_1)$ and $Y = (Q_2, v_2)$ in $\xi_X(R_X)$, $\tau_X(Z) = Z'$ and $\tau_X(Y) = Y'$, we have a commutative diagram,*

$$\begin{array}{ccccccc} 0 \to & _{\xi_X(R_X)}(Z,Y) & \xrightarrow{\sigma_{Z,Y}} & _{R_X}(Q_1,Q_2) & \xrightarrow{\delta_{Z,Y}} \\ & \downarrow^{\tau_X} & & \downarrow^{\phi_1} & \\ 0 \to & _{\xi(R)}(Z',Y') & \xrightarrow{\sigma_{Z',Y'}} & _R(P \otimes_{R_X} Q_1, P \otimes_{R_X} Q_2) & \xrightarrow{\delta_{Z',Y'}} \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\delta_{Z,Y}} & _{R_X}(Q_1, M_X \otimes_{R_X} Q_2) & \xrightarrow{\eta_{Z,Y}} & \mathrm{Ext}_{\xi_X(R_X)}(Z,Y) & \to 0 \\ & \downarrow^{(i_N \otimes 1)_* \phi_2} & & \downarrow^{\psi_X} & \\ \xrightarrow{\delta_{Z',Y'}} & _R(P \otimes_{R_X} Q_1, M \otimes_R P \otimes_{R_X} Q_2) & \xrightarrow{\eta_{Z',Y'}} & \mathrm{Ext}_{\xi(R)}(Z',Y') & \to 0 \end{array}$$

where ψ_X is the monomorphism of $\mathrm{End}_{\xi_X(R_X)}(Y) - \mathrm{End}_{\xi_X(R_X)}(Z)$-bimodules induced by τ_X.

Theorem (4.6). [7] *For any pair of objects $Z, Y \in \xi_X(R_X)$ there is a short exact sequence of $\mathrm{End}_{\xi_X(R_X)}(Y) - \mathrm{End}_{\xi_X(R_X)}(Z)$-bimodules $(\tau_X(?) = ?')$*

$$\mathrm{Ext}_{\xi_X(R_X)}(Z,Y) \overset{\psi_X}{\to} \mathrm{Ext}_{\xi(R)}(Z',Y') \overset{\psi_N}{\to} \mathrm{Ext}_{\xi_N(R_N)}(F_N(Z'), F_N(Y')) .$$

5. Tensor product and exact structures

In this section R is a k-algebra, where k is a field.

Lemma (5.1). *Let R and S be k-algebras, M an R-module and P a finitely generated projective R-module. There is an isomorphism of $S-S$-bimodules*

$$\varphi_{P,M} : \mathrm{Hom}_R(P,M) \otimes_k S \to \mathrm{Hom}_{R \otimes_k S}(P \otimes_k S, M \otimes_k S)$$

natural in P and M.

Let Q be a finitely generated projective R-module, $f \otimes s_1 \in \mathrm{Hom}_R(P,Q) \otimes_k S$ and $g \otimes s_2 \in \mathrm{Hom}_R(Q,M) \otimes_k S$, then

$$(\varphi_{Q,M}(g \otimes s_2))(\varphi_{P,Q}(f \otimes s_1)) = \varphi_{P,M}(gf \otimes s_1 s_2)$$

Now let $\alpha : S \to T$ be a morphism of k-algebras and consider the canonical isomorphism of $R \otimes_k T$-bimodules $(R \otimes_k T) \otimes_{R \otimes_k S} (P \otimes_k S) \cong P \otimes_k T$, and the *functor* $(R\otimes_k T)\otimes_{(R\otimes_k S)}(?) : R\otimes_k S-\mathrm{Mod}\to R\otimes_k T-\mathrm{Mod}$, *then we have the commutative diagram*

$$\begin{array}{ccc}\mathrm{Hom}_R(P,M)\otimes_k S & \xrightarrow{1\otimes\alpha} & \mathrm{Hom}_R(P,M)\otimes_k T\\ \downarrow\varphi^S_{P,M} & & \downarrow\varphi^T_{P,M}\\ \mathrm{Hom}_{R\otimes_k S}(P\otimes_k S, M\otimes_k S) & \xrightarrow{(R\otimes_k T)\otimes_{(R\otimes_k S)}(?)} & \mathrm{Hom}_{R\otimes_k T}(P\otimes_k T, M\otimes_k T)\end{array}$$

Proof. Let $\rho : S\to \mathrm{Hom}_S(S,S)^{op}$ be the canonical isomorphism. Let φ be the composition of the isomorphisms of $S-S$-bimodules natural in P and M

$$\mathrm{Hom}_R(P,M)\otimes_k S\to \mathrm{Hom}_R(P,M\otimes_k S)\to \mathrm{Hom}_{R\otimes_k S}(P\otimes_k S, M\otimes_k S)\ .$$

This composition sends the element $f\otimes y$ to $f\otimes\rho_y$.

The rest of the claim is immediate. □

Remark (5.2). A lift pair (ξ,R) and a k-algebra S determine a new lift pair $\left(\xi^S, R\otimes_k S\right)$ given by

$$\xi^S:\quad 0\to\quad M\otimes_k S\quad \xrightarrow{i\otimes 1} E\otimes_k S\quad \xrightarrow{\pi\otimes 1}\quad R\otimes_k S\quad\to 0$$

where $?\otimes_k S$ is an $R\otimes_k S$-bimodule with action $(r_1\otimes s_1)(x\otimes s)(r_2\otimes s_2) = r_1xr_2\otimes s_1ss_2$.

We are mainly interested in the full subcategory $(\xi(R))^S$ of the objects $(P\otimes_k S, e)$, where P is a finitely generated projective R-module. By the isomorphism of $R\otimes_k S$-bimodules $(E\otimes_k S)\otimes_{R\otimes_k S}(P\otimes_k S)\cong E\otimes_R P\otimes_k S$, we are going to consider that $e: P\otimes_k S\to E\otimes_R P\otimes_k S$. Then, by Lemma (5.1), the exactness of $\_\otimes_k S$ and an argument similar to Remark (2.1)

$$e=\nu_\omega\otimes 1+\sum_j (i\otimes 1\otimes 1)\left(f_j\otimes\rho_{s_j}\right)$$

for $\omega\in E$ such that $\pi(\omega)=1$, and $f_i\in\mathrm{Hom}_R(P, M\otimes_R P)$.

If $\alpha: S\to T$ is a morphism of k-algebras, then $1\otimes\alpha$ induces a morphism of lift categories $(1_E\otimes\alpha, 1_R\otimes\alpha):\xi^S(R\otimes_k S)\to\xi^T(R\otimes_k T)$ and a functor $F_\alpha:\xi^S(R\otimes_k S)\to\xi^T(R\otimes_k T)$, which in morphisms and underlying modules is the functor $(R\otimes_k T)\otimes_{(R\otimes_k S)}(?)$. We are going to denote by $G_\alpha:(\xi(R))^S\to(\xi(R))^T$ the restriction of F_α to $(\xi(R))^S$.

In the particular case of the canonical morphism $\alpha: k\to S$ we denote by F^S the composition of functors $\xi(R)\cong(\xi(R))^k\overset{G_\alpha}{\to}(\xi(R))^S$. Let us observe that $F^S((P,e))=(P\otimes_k S, e\otimes 1)$ and $F^S(f)=f\otimes 1$.

Proposition (5.3). *Let $\xi(R)$ be a lift category and $\alpha: S\to T$ a morphism of k-algebras.*

1. *Let $X=(P_1\otimes_k S, e_1)$ and $Y=(P_2\otimes_k S, e_2)$ be objects in $(\xi(R))^S$. There is a morphism of* $\mathrm{End}\left(F^S(Y)\right)-\mathrm{End}\left(F^S(X)\right)$*-bimodules* $\underline{G}^\alpha:\mathrm{Ext}_{\xi^S}(X,Y)\to\mathrm{Ext}_{\xi^T}(G_\alpha(X),G_\alpha(Y))$. *If α is surjective then $\underline{G}^\alpha$ is surjective.*
2. *Let X,Y be objects in $\xi(R)$, and S a k-algebra. There are canonical isomorphisms of* $\mathrm{End}\left(F^S(Y)\right)-\mathrm{End}\left(F^S(X)\right)$*-bimodules*

$$\mathrm{Hom}_{(\xi(R))^S}\left(F^S(X),F^S(Y)\right)\cong\mathrm{Hom}_\xi(X,Y)\otimes_k S^{op}$$

$$\mathrm{Ext}_{(\xi(R))^S}\left(F^S(X),F^S(Y)\right)\cong\mathrm{Ext}_\xi(X,Y)\otimes_k S^{op}.$$

Proof. For Remark (5.2) there is a commutative diagram ($G_\alpha(X) = X'$, $G_\alpha(Y) = Y'$)

$$\begin{array}{ccc} {}_{R\otimes_k S}(P_1 \otimes_k S, P_2 \otimes_k S) & \overset{\delta_{X,Y}}{\longrightarrow} & {}_{R\otimes_k S}(P_1 \otimes_k S, M \otimes_R P_2 \otimes_k S) \\ \downarrow \text{-}\otimes_S T & & \downarrow \text{-}\otimes_S T \\ {}_{R\otimes_k T}(P_1 \otimes_k T, P_2 \otimes_k T) & \overset{\delta_{X',Y'}}{\longrightarrow} & {}_{R\otimes_k T}(P_1 \otimes_k T, M \otimes_R P_2 \otimes_k T) \end{array}$$

From this diagram it follows the first claim.

The second claim is true by the exactness of the functor $\_ \otimes_k S^{op}$ and the commutative diagram

$$\begin{array}{ccc} {}_{R\otimes_k S}(P_1 \otimes_k S, P_2 \otimes_k S) & \overset{\delta_{F^S(X),F^S(Y)}}{\longrightarrow} & {}_{R\otimes_k S}(P_1 \otimes_k S, M \otimes_R P_2 \otimes_k S) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_R(P_1, P_2) \otimes_k S^{op} & \overset{\delta_{X,Y}\otimes 1}{\longrightarrow} & \mathrm{Hom}_R(P_1, M \otimes_R P_2) \otimes_k S^{op} \end{array}$$

□

The goal of this section is to show that for a lift pair (R, ξ) and a k-algebra S, we can consider the behavior of the lift category $(\xi(R))^S$ as componentwise-like. The next proposition reinforces this idea; if we have a functor of lift categories $F : \xi_1(R_1) \to \xi_2(R_2)$, then by $F \otimes 1 : (\xi_1(R_1))^S \to (\xi_2(R_2))^S$ we mean the functor given by $(F \otimes 1)(P \otimes_k S, e) = (P' \otimes_k S, e')$ where $e = \nu_\omega \otimes 1 + \sum_j (i \otimes 1 \otimes 1)\left(f_j \otimes \rho_{s_j}\right)$ and $e' = F(\nu_\omega) \otimes 1 + \sum_j F((i \otimes 1) f_j) \otimes \rho_{s_j}$, and on morphisms by $(F \otimes 1)(\sum_l g_l \otimes \rho_{s_l}) = (\sum_l F(g_l) \otimes \rho_{s_l})$.

Proposition (5.4). *Let $\phi : (R_1, \xi_1) \to (R_2, \xi_2)$ be a morphism of lift pairs, and $\alpha : S \to T$ a morphism of k-algebras. Let $F_\phi : \xi_1(R_1) \to \xi_2(R_2)$, $F^i_{1\otimes\alpha} : \xi^S_i(R_i \otimes_k S) \to \xi^T_i(R_i \otimes_k T)$ and $G^i_\alpha : (\xi_i(R_i))^S \to (\xi_i(R_i))^T$ be the induced functors.*

1. *There is a commutative diagram of functors*

$$\begin{array}{ccc} (\xi_1(R_1))^S & \overset{(F_\phi)\otimes 1}{\longrightarrow} & (\xi_2(R_2))^S \\ \downarrow G^1_\alpha & & \downarrow G^2_\alpha \\ (\xi_1(R_1))^T & \overset{(F_\phi)\otimes 1}{\longrightarrow} & (\xi_2(R_2))^T \end{array}$$

In particular, if N is a sub-bimodule of M and J is the radical of R, we have $(F_N \otimes 1) G_\alpha = G'_\alpha (F_N \otimes 1)$ and $(F_J \otimes 1) G_\alpha = G'_\alpha (F_J \otimes 1)$.

2. *Let N be a sub-bimodule of M and $X = (P, e)$ a finite object in $\xi_N(R)$. There is a commutative diagram of functors*

$$\begin{array}{ccc} (\xi_X(R_X))^S & \overset{(\tau_X)\otimes 1}{\longrightarrow} & (\xi(R))^S \\ \downarrow G^X_\alpha & & \downarrow G_\alpha \\ (\xi_X(R_X))^T & \overset{(\tau_X)\otimes 1}{\longrightarrow} & (\xi(R))^T \end{array}$$

Proof. It is not hard to prove the commutativity of the diagram

$$\begin{array}{ccc} \xi^S_1(R_1 \otimes_k S) & \overset{F_{(\phi\otimes 1)}}{\longrightarrow} & \xi^S_2(R_2 \otimes_k S) \\ \uparrow & & \uparrow \\ (\xi_1(R_1))^S & \overset{(F_\phi)\otimes 1}{\longrightarrow} & (\xi_2(R_2))^S \end{array}$$

where the vertical arrows are the canonical embeddings. This and the evident identity $F_{\phi\otimes 1}F^1_{1\otimes\alpha} = F^2_{1\otimes\alpha}F_{\phi\otimes 1}$ provide the proof for the first statement.

The last claim can be proved through easy computations. □

Remark (5.5). Only for this remark we denote, for a left R-module L, ${}^*L = \mathrm{Hom}_R(L,R)$, and for a right R-module M, $M^* = \mathrm{Hom}_R(M,R)$.

Now suppose E_R is projective finitely generated; then there are natural isomorphisms

${}_R(P, E\otimes_R P) \to_R (P, {}^*(E^*)\otimes_R P) \to_R (P, {}_R(E^*, P)) \to_R (E^*\otimes_R P, P)$.

Let $\psi : {}_R(P, E\otimes_R P) \to_R (E^*\otimes_R P, P)$ be the composition above, (μ_i, x_i) a right dual base of E_R and $\phi : R\otimes_R P \to P$ the canonical isomorphism. Then for $f \in_R (P, E\otimes_R P)$ and $\mu\otimes p \in E^*\otimes_R P$, we have $\psi(f)(\mu\otimes p) = \phi(\mu\otimes 1) f(p)$, and for $g\in_R (E^*\otimes_R P, P)$ and $p\in P$ we have $\psi^{-1}(g)(p) = \sum_i x_i\otimes g(\mu_i\otimes p)$.

Observe that $(\mu_i\otimes Id_S, x_i\otimes 1)$ is a right dual base for $(E\otimes_k S)_{R\otimes_k S}$, and that there is a natural isomorphism

$\psi_0 : {}_{R\otimes_k S}(P\otimes_k S, (E\otimes_k S)\otimes_{R\otimes_k S}(P\otimes_k S)) \to$
${}_{R\otimes_k S}\left((E\otimes_k S)^*\otimes_{R\otimes_k S}(P\otimes_k S), (P\otimes_k S)\right)$.

By Lemma (5.1), $f\in_{R\otimes_k S}(P\otimes_k S, (E\otimes_k S)\otimes_{R\otimes_k S}(P\otimes_k S))$ corresponds to $\sum f_j\otimes\rho_{s_j}$ where $f_j\in_R (P, E\otimes_R P)$ and $\rho_{s_j}\in_S (S,S)$ is right multiplication by s_j. In a similar way we have for $\theta\in(E\otimes_k S)^*$ that it can be written as $\sum_t(\mu_t\otimes\lambda_{s_t})$, where $\mu_t\in E^*$ and $\lambda_{s_t}\in(S,S)_S$ is left multiplication by s_t. An easy computation shows

$$\psi_0(f)(\theta\otimes p\otimes s) = \sum_{j,t}\phi((\mu_t\otimes 1) f_j(p))\otimes s_t s s_j\,.$$

Then e determines an object $(P\otimes_k S, e)$ of the lift pair $\left(\xi^S, R\otimes_k S\right)$ if and only if $\psi_0(e)(\pi\otimes 1\otimes p\otimes s) = p\otimes s$.

Let be $A_\xi = T_R(E^*)/(\pi - 1)$, a notation similar to the one used in [6]. In [9] it is proved that there is an equivalence of categories $\psi : \xi(R) \to P_R A_\xi$, where $P_R A_\xi$ is the full subcategory of left A_ξ- modules which are projectives as R-modules. The argument above proves, in a similar way, that there is an equivalence of categories $\psi_S\colon\ (\xi(R))^S \xrightarrow{\cong} P_R A_{\xi^S}$ where $P_R A_{\xi^S}$ is the full subcategory of the modules of the k-algebra $A_{\xi^S} = T_{R\otimes_k S}\left((E\otimes_k S)^*\right)/(\pi\otimes 1 - 1\otimes 1)$ of the form $P\otimes_k S$.

Moreover, there is a canonical isomorphism of $R\otimes_k S$-algebras $\gamma_0 : T_{R\otimes_k S}\left((E\otimes_k S)^*\right) \to T_R(E^*)\otimes_k S$ given by $\gamma_0(\mu_1\otimes\lambda_{s_1}\otimes ...\otimes\mu_n\otimes\lambda_{s_n}) = (\mu_1\otimes ...\otimes\mu_n)\otimes s_1...s_n$, which induces an isomorphism

$$\gamma : A_{\xi^S} \to (T_R(E^*)\otimes_k S)/(\pi\otimes 1 - 1\otimes 1) \cong A_\xi\otimes_k S\,.$$

Let ψ_f denotes the action of the algebra A_{ξ^S} on $P\otimes_k S$ induced by $\psi_0(f)$. The formulas above prove, for $\mu\in E^*$ and $f\cong\sum f_j\otimes\rho_{s_j}$ that

$$\psi_f\left(\gamma^{-1}(\mu\otimes s_0)\otimes p\otimes s\right) = \sum_j\phi((\mu\otimes 1) f_j(p))\otimes s_0 s s_j\,.$$

Let $\alpha : S\to T$ be a morphism of k-algebras, and $H_\alpha : P_R A_{\xi^S} \to P_R A_{\xi^T}$ the functor induced by $1\otimes\alpha : A_\xi\otimes_k S \to A_\xi\otimes_k T$. The last formula proves the next claim.

Proposition (5.6). *Let (R,ξ) be a lift pair with E_R projective finitely generated, and $\alpha : S \to T$ a morphism of k-algebras. The following diagram commutes,*

$$\begin{array}{ccc} (\xi(R))^S & \overset{\psi_S}{\to} & P_R A_{\xi^S} \\ \downarrow^{G_\alpha} & & \downarrow^{H_\alpha} \\ (\xi(R))^T & \overset{\psi_T}{\to} & P_R A_{\xi^T} \end{array}$$

where H_α is equivalent to the functor $\left(A_{\xi^T} \otimes_k T\right) \otimes_{\left(A_{\xi^S}\otimes_k S\right)} ?$.

Remark (5.7). The equivalences $\xi(R) \cong P_R A_\xi$ and $(\xi(R))^S \overset{\psi_S}{\to} P_R A_{\xi^S}$ send conflations to short exact sequences; moreover, they induce isomorphisms of bimodules between extensions.

6. Parametrizations of objects with selfextensions of dimension one, when k is algebraically closed

Definition (6.1). Let $X = (P, e)$ be a finite object of $\xi(R)$. The norm of X is defined as

$$\|X\| = \dim_k \operatorname{Hom}_R(P, M \otimes_R P) .$$

Lemma (6.2). [6] *Let $X_1, ..., X_n$ be nonisomorphic indecomposable finite objects of $\xi_N(R)$. Denote $X = \oplus_i X_i$, and by h_i the primitive idempotent determined by X_i in $R_X = \operatorname{End}_{\xi_N(R)}(X)^{op}$. Consider the lift pair (ξ_X, R_X) and an object $Z = (Q, v)$ in $\xi_X(R_X)$. Then $Q \cong \oplus m_i R_X h_i$ if and only if $F_N \tau_X(Z) \cong \oplus m_i X_i$.*

Remark (6.3). In [7] it is proved, for the reduction functors $F_N : \xi(R) \to \xi_N(R)$ and $\tau_X : \xi_X(R_X) \to \xi(R)$, that if $Y = (Q, g)$ and $Z = (P, e)$ are finite objects of $\xi_X(R_X)$ and $\xi(R)$ respectively, then $\|Z\| \geq \|F_N(Z)\|$ and $\|\tau_X(Y)\| \geq \|Y\|$. If any indecomposable projective R-module is isomorphic to a direct summand of P we will say that P is *a-sincere.* If P is a-sincere and $M \otimes_R P \neq 0$, the first inequality is strict. Also, the second inequality is strict if P is a-sincere and $M/N \neq 0$.

As is shown in [6] it is possible to reduce the discussion on some categorical property of $\xi(R)$ to the case of a basic ring. Consider $M = N$ and the induced exact sequence

$$\xi_N : \quad 0 \longrightarrow 0 \longrightarrow \frac{E}{M} \overset{\pi_N}{\longrightarrow} R \longrightarrow 0$$

Then the forgetful functor $F_0 : \xi_N(R) \to R$-Proj is an equivalence of categories. Take the finite objects $X_1, ..., X_n$ of $\xi_N(R)$ corresponding to the non-isomorphic indecomposable projectives $P_1, ..., P_n$ in $\operatorname{mod} R$. If we make $X = X_1 \oplus ... \oplus X_n$, we obtain a basic algebra $R_X = \operatorname{End}_{\xi_N}(X)^{op}$, the corresponding lift pair (R_X, ξ_X), and an equivalence of categories $\xi_X(R_X) \overset{\tau_X}{\to} \xi(R)$. Moreover, if we are interested in a subcategory of finite objects with P fixed up to isomorphism, then we can select $X = \oplus_i \left(P_i, \nu_1^i\right)$ where the P_i are the non-isomorphic indecomposable projectives in $\operatorname{mod} R$ which are a direct summand of P. Then, by Lemma (6.2), the corresponding subcategory of $\xi_X(R_X)$, up to isomorphism has a fixed, a-sincere Q.

Lemma (6.4). *Let $\xi : M \to E \to R$ be a lift pair with R a basic finite dimensional k-algebra, k algebraically closed, M a simple R-R-bimodule and $\psi : \xi_J(R/J) \to A_{\xi_J}$ as in Remark (5.5). Then A_{ξ_J} is isomorphic to one of the following types of algebras:*

1. $k \times ... \times k \times k$,
2. $k \times ... \times k \times \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$,
3. $k \times ... \times k \times k[x]$.

In the last two cases, the R-morphism $f : P \to P'$ is a morphism in $\xi(R)$ from the object $X = (P, e)$ to the object $Y = (P', e')$, if and only if the induced R/J-morphism $f^J : P/JP \to P'/JP'$ is a morphism from $\psi F_J(X)$ to $\psi F_J(Y)$.

Proof. By Lemma (2.7), $JM = MJ = 0$ and then M is a simple R/J-R/J-bimodule. We have $R/J \cong k \times ... \times k$, so $\dim_k M = 1$.

If $M \cap (JE + EJ) = M$, the functor F_J annihilates M, and the first case follows.

If $M \cap (JE + EJ) = 0$, then the last two cases follow by Lemma 8.5 of [6], and formula (2.2) shows that $e'f - (1 \otimes f)e = F_J(e') f^J - (1 \otimes f^J) F_J(e')$. □

Definition (6.5). Let $\mathcal{F}$ be a family of objects in $\xi(R)$ closed under isomorphisms. Let $W = (P \otimes_k k[x], e) \in (\xi(R))^{k[x]}$ and $G_{\alpha(\lambda)} : (\xi(R))^{k[x]} \to \xi(R)^k = \xi(R)$ the functor induced by the evaluation $\alpha(\lambda) : k[x] \to k$ for any $\lambda \in k$. We will say that $\mathcal{F}$ is parametrized by W if

1. For almost all $\lambda \in k$, $G_{\alpha(\lambda)}(W) \in \mathcal{F}$,
2. For almost all isomorphism class $[X] \in \mathcal{F}$ there exists $\lambda \in k$ such that $G_{\alpha(\lambda)}(W) \cong X$.

In this case we will say that $\mathcal{F}$ is one-parametrized.

For algebras we will use a similar concept.

Lemma (6.6). *Let $\xi(R)$ be a lift category with R a finite dimensional k-algebra and k an arbitrary field. Let $\mathcal{F}$ be a family of objects parametrized by $W = (P \otimes_k k[x], e)$. Then the k-dimension of the selfextensions of $G_{\alpha(\lambda)}(W)$ is a fixed n for almost all $\lambda \in k$, and, in general, $dim_k \left(G_{\alpha(\mu)}(W)\right) \leq n$ for $\mu \in k$.*

Proof. Let $W_\lambda = G_{\alpha(\lambda)}(W)$. By Remark (5.2) we can consider $e = \nu_\omega \otimes 1 + \sum_j (i \otimes 1 \otimes 1)\left(f_j \otimes \rho_{s_j}\right)$. Then we have a commutative diagram

$$\begin{array}{ccc} {}_R(P,P) \otimes_k k[x] & \xrightarrow{\delta_{W,W}} & {}_R(P, M \otimes_R P) \otimes_k k[x] \\ \downarrow{\scriptstyle 1 \otimes \frac{k[x]}{\langle x-\lambda \rangle}} & & \downarrow{\scriptstyle 1 \otimes \frac{k[x]}{\langle x-\lambda \rangle}} \\ {}_R(P,P) & \xrightarrow{\delta_{W_\lambda, W_\lambda}} & {}_R(P, M \otimes_R P) \end{array}$$

$\delta_{W,W}$ is a morphism between free finitely generated $k[x]$-modules, so its image has a base, and the rank of this is, for almost all $\lambda \in k$, the same as the dimension of the image of $\delta_{W_\lambda, W_\lambda}$. □

Proposition (6.7). *Let $\xi(R)$ be a lift category with R a finite dimensional k-algebra and k algebraically closed. Let $\mathcal{F}$ be a family of objects parametrized by $W = (P \otimes_k k[x], e)$. Then the set of isoclasses of indecomposable objects in $\mathcal{F}$ is finite or cofinite.*

Proof. $\xi(R)$ is equivalent to the category of objects (P,∇) where P is an R-projective module y $\nabla : P \to M \otimes_R P$ is a k-linear function such that $\nabla(rp) = \Delta(r) \otimes p + r\nabla(p)$ for all $r \in R$, $p \in P$ and the morphisms $f : (P,\nabla) \to (P',\nabla')$ are R-morphisms $f : P \to P'$ such that $(1 \otimes f)\nabla = \nabla' f$ [9].

P and $M \otimes_R P$ are k-spaces of finite dimension, so we can associate matrices to the action of the morphisms and the multiplication by R. Then it is easy to see that the set of matrices corresponding to the pairs (∇, f) where f is invertible is constructible in the Zariski topology. The subset of pairs such that $(1 \otimes f)\nabla f^{-1}$ is a block matrix is constructible, so the image under the projection $(\nabla, f) \to \nabla$ is a constructible set. The complement of this image corresponds to the indecomposable objects. □

Definition (6.8). Let $\xi(R)$ be a lift category. By $H_\xi(P,n)$ we will denote the full subcategory of indecomposable objects $Z = (P, e_Z)$, where P is fixed up to isomorphism and $dim_k \operatorname{Ext}_\xi(Z,Z) = n$. The number $\|H_\xi(P,n)\| = \|Z\|$, for any $Z \in H_\xi(P,n)$, will be called the norm of this subcategory.

Definition (6.9). Let H_1 and H_2 be families in the categories $C_1$and C_2 respectively, and let $F : C_1 \to C_2$ be a functor. We will say that H_1 covers H_2 via F if:

1. For any $X \in C_1$ there exists $Y \in C_2$ such that $F(X) \cong Y$
2. For almost all isomorphism class $[Y] \in H_2$ there exists $X \in C_1$ such that $F(X) \cong Y$.

Theorem (6.10). *Let $\xi(R)$ be a lift category with R a finite dimensional k-algebra and k algebraically closed.*

1. *Let $X = (P_1, e_1)$ and $Y = (P_2, e_2)$ be finite objects in $\xi(R)$ such that $\operatorname{Ext}_{\xi(R)}(X,X) = 0$ and $\operatorname{Ext}_{\xi(R)}(Y,Y) = 0$. Then $X \cong Y$ if and only if $P_1 \cong P_2$.*
2. *$H_\xi(P,1)$ has a finite number of isoclasess or is one-parametrized.*

Proof. The first claim is a particular case of Theorem 5.1 of [7].

Suppose $H_\xi(P,1)$ has infinite isoclasses. By Theorem (4.6), Lemma (6.2) and Remark (6.3) we can obtain a functor $\tau_X : \xi_X(R_X) \to \xi(R)$ with R_X basic and such that $H_{\xi_X}(Q,1)$ is sent to $H_\xi(P,1)$, where Q is fixed and a-sincere, so in the rest of the proof we will assume R basic and P a-sincere.

The proof will be done by induction on the norm of $H_\xi(P,1)$ so assume the claim proved for any $H_{\xi'}(P',1)$ with norm less than $n = \|H_\xi(P,1)\|$.

Let N be a maximal R-R-sub-bimodule of M, $F_N : \xi(R) \to \xi_N(R)$ and $F_J : \xi_N(R) \to \xi_{NJ}(R/J)$ the associated functors and $\psi : \xi_{NJ}(R/J) \to A_\xi$ the equivalence of categories of Remark (5.5).

First assume A_ξ is of finite representation type. Then we choose $X = \oplus_i^s X_i$ where $(X_1, ..., X_s)$ is a complete and irreducible system of isoclasses of indecomposable objects in $\xi_N(R)$.

Let

$$\mathcal{F}_i = \{Z \in H_\xi(P,1) | \dim_k \operatorname{Ext}_{\xi_N(R)}(F_N(Z), F_N(Z)) = i\},$$

so $H_\xi(P,1) = \mathcal{F}_0 \cup \mathcal{F}_1$.

For $Z \in \mathcal{F}_1$ we have $F_N(Z) \cong \oplus_i^n m_i' X_i$ where the numbers m_i' are independent of Z, only depending on P. By Lemma (6.2) and Theorem (4.6), the subcategory

$H_{\xi_X(R_X)}(\oplus_i^n m_i' R_X h_i, 0)$ covers $\mathcal{F}_1$ via $\tau_X : \xi_X(R_X) \to \xi(R)$; it follows from the first part of the statement that $\mathcal{F}_1$ has one isoclass.

If $Z \in \mathcal{F}_0$ we have $F_N(Z) \cong \oplus_i^n m_i X_i$ and as before the numbers m_i only depend of P. We have that the subcategory $H_{\xi_X(R_X)}(\oplus_i^n m_i R_X h_i, 1)$ covers $\mathcal{F}_0$ via $\tau_X : \xi_X(R_X) \to \xi(R)$; the first subcategory is parametrized by the hypothesis of induction, and by Proposition (5.4) $\mathcal{F}_0$ is parametrized, so $H_\xi(P,1)$ is parametrized.

Now suppose that A_ξ is of infinite representation type. By the a-sincereness of P and Lemma (6.4) it follows, for any $Z \in H_\xi(P,1)$, that
$dim_k\left(\mathrm{Ext}_{\xi_N(R)}(F_N(Z), F_N(Z))\right) = 1$ and the vector dimension of $\psi F_J F_N(Z)$ has fixed dimension $(m_1, m_2, ..., m_{s-1}, 1)$ where the last number corresponds to a simple $k[x]$-module. We denote $X^\lambda = \left(\oplus_i^{s-1} X_i\right) \oplus X_\lambda$ where $(X_1, ..., X_{s-1})$ is a complete and irreducible system of isoclasses of indecomposable objects in $\xi_N(R)$ with trivial selfextensions, and X_λ corresponds to the simple $k[x]$-module in A_ξ where the action of x is multiplication by $\lambda \in k$.

Let $g : P \to M \otimes_R P$ be such that $F_N(P, \nu_\omega + \lambda g) \in \mathrm{add}\, X^\lambda$, and $u : P \to N \otimes_R P$ such that for some μ the object $(P, \nu_\omega + \mu g + u)$ is in $H_\xi(P,1)$. It follows by Lemma (6.6) and Theorem (4.6) that the dimension of the selfextensions $W_\lambda = (P, \nu_\omega + \lambda g + u)$ is one for almost all λ.

For the first part of the statement we have that in $\xi_{X^\lambda}(R_{X^\lambda})$ there is at most one isoclass of an object with underlying projective isomorphic to $\oplus_i^s m_i R_{X^\lambda} h_i$ and with trivial group of selfextensions; then for any λ there is at most one isoclass, let us call it Z_λ, in $H_\xi(P,1)$ such that $F_N(Z_\lambda) \in \mathrm{add}\, X^\lambda$. Moreover, $W_\lambda \cong Z_\lambda$ for infinite λ, so by Proposition (6.7) for almost all λ we have $W_\lambda \in H_\xi(P,1)$, so this subcategory it is parametrized by

$$(P \otimes_k k[x], \nu_\omega \otimes 1 + g \otimes x + u \otimes 1)\ . \qquad \square$$

7. Lift categories and finite dimensional algebras

Now we apply the previous result to the category of f.g. modules on a finite dimensional algebra.

Definition (7.1). Let Λ be an Artin algebra. $P^1(\Lambda)$ is the category with objects the morphisms $f : P_1 \to P_0$ with $\mathrm{Im}\, f \subset \mathrm{rad}\, P_0$. A morphism from the object $f : P_1 \to P_0$ to the object $g : Q_1 \to Q_0$ is a pair of R-morphisms $u_i : P_i \to Q_i$ such that $u_0 f = g u_1$. $P^2(\Lambda)$ is the full subcategory of $P^1(\Lambda)$ with objects $f : P_1 \to P_0$ such that $\mathrm{Ker}\, f \subset \mathrm{rad}\, P_1$.

Definition (7.2). Let Λ be an Artin algebra. The associated lift pair $\left(R^\Lambda, \xi^\Lambda\right)$, where R^Λ is the matrix algebra $\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$ and $J = \mathrm{rad}\, \Lambda$, is the exact short sequence of R^Λ-R^Λ-bimodules

$$\xi^\Lambda : 0 \to \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} \Lambda & J \\ 0 & \Lambda \end{pmatrix} \to \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \to 0\,.$$

In 1.7 of [9] an equivalence $G : \xi^\Lambda\left(R^\Lambda\right) \to P^1(\Lambda)$ is given as follows: let $Y = ((P_1, P_0), e)$ be an object in $\xi^\Lambda\left(R^\Lambda\right)$. The isomorphism

$$\begin{pmatrix} \Lambda & J \\ 0 & \Lambda \end{pmatrix} \otimes_{R^\Lambda} \begin{pmatrix} P_1 \\ P_0 \end{pmatrix} \cong \begin{pmatrix} P_1 \oplus JP_0 \\ P_0 \end{pmatrix}$$

shows that e is determined by $1_{(P_1,P_0)}$ and a Λ-morphism $f: P_1 \to JP_0$, so $G(Y) = if$, where $i: JP_0 \to P_0$ is the inclusion, and G is the identity in morphisms.

Also there is a bijection between the conflations in $\xi^{\Lambda}(R^{\Lambda})$ and the pairs $((u_0, u_1), (v_0, v_1))$ of composables morphisms in $P^1(\Lambda)$, such that $P_1 \overset{u_0}{\to} P_1' \overset{v_0}{\to} P_1''$ and $P_0 \overset{u_1}{\to} P_0' \overset{v_1}{\to} P_0''$ are short exact sequences, then we define the conflations in $P^1(\Lambda)$ as those kind of composable pairs, so we have an exact structure by section 3.

Corollary (7.3). [3] *Let $f_L : Q_L \to P_L$ and $f_N : Q_N \to P_N$ be objects of $P^1(\Lambda)$. There is an exact sequence*

$$\begin{array}{llll} 0 \to & \mathrm{Hom}_{P^1(\Lambda)}(f_N, f_L) \overset{\sigma_{N,L}}{\longrightarrow} & \mathrm{Hom}_{\Lambda}(Q_N, Q_L) \oplus \mathrm{Hom}_{\Lambda}(P_N, P_L) & \overset{\delta_{N,L}}{\longrightarrow} \\ & \mathrm{Hom}_{\Lambda}(Q_N, JP_L) \overset{\eta_{N,L}}{\longrightarrow} & \mathrm{Ext}_{P^1(\Lambda)}(f_N, f_L) & \to 0 \end{array}$$

where $\sigma_{N,L}$ is the canonical inclusion, $\delta_{N,L}(u,v) = vf_N - f_L u$ and $\eta_{N,L}(g)$ is the object $\begin{pmatrix} f_L & g \\ 0 & f_N \end{pmatrix} : Q_L \oplus Q_N \to P_L \oplus P_N$.

Proof. This is a consequence of Proposition (3.6) using the equivalence G. ☐

Proposition (7.4). [5] *Let L, N be Λ-modules and $Q_L \overset{f_L}{\to} P_L \overset{\pi_L}{\to} L$ and $Q_N \overset{f_N}{\to} P_N \overset{\pi_N}{\to} N$ the corresponding minimal projective presentations. Then*

$$\dim_k \mathrm{Ext}_{P^1(\Lambda)}(f_N, f_L) = \dim_k \mathrm{Hom}_{\Lambda}(L, DTrN)/S(L, DTrN).$$

where $S(L, DTrN)$ is the subspace of morphisms factorizing through semisimple Λ-modules.

Definition (7.5). Let $f: Q \to P$ be an element of $P^1(\Lambda)$. Then we say that $\underline{\dim}(f) = (n_1, ..., n_j, n_1', ..., n_j')$, when $Q \cong \oplus_{i=1}^{j} n_i P_i$, $P \cong \oplus_{i=1}^{j} n_i' P_i$ and $(P_1, ..., P_j)$ is a complete and irreducible system of representatives of the isoclasses of indecomposables projectives Λ-modules.

We will denote by $F((n_1, ..., n_j, n_1', ..., n_j'), m)$ the family of indecomposables Λ-modules M such that $\underline{\dim}(f_M) = (n_1, ..., n_j, n_1', ..., n_j')$ and

$$\dim_k \mathrm{Hom}_{\Lambda}(M, DTrM)/S(M, DTrM) = m.$$

Corollary (7.6). *Let Λ be a finite dimensional k-algebra, with k algebraically closed. $F((n_1, ..., n_j, n_1', ..., n_j'), 1)$ has a finite number of isoclasses or is one-parametrized.*

Proof. By Theorem (6.10) $H((n_1, ..., n_j, n_1', ..., n_j'), 1)$ has a finite number of isoclasses or is one-parametrized. Then, by Lemma 6.1 of [6] the claim follows. ☐

Example (7.7). Let Λ be the wild k-algebra (k algebraically closed)

$$\circ_1 \ \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \ \circ_2 \overset{\gamma}{\to} \circ_3 .$$

We will denote $P_i = \Lambda e_i$. In this example we are interested in the morphisms $P_3 \to JP_1$, corresponding to indecomposable objects in $P_1(\Lambda)$, such that the dimension of the selfextensions is one. By Definition (7.2) we are just looking for the indecomposable objects

$$\begin{pmatrix} P_3 \\ P_1 \end{pmatrix} \to \begin{pmatrix} \Lambda & J \\ 0 & \Lambda \end{pmatrix} \otimes \begin{pmatrix} P_3 \\ P_1 \end{pmatrix}$$

in the associated lift category $\xi(R(\Lambda))$ with space of selfextensions of dimension one.

We can save steps taking $N = \begin{pmatrix} 0 & \langle \gamma\alpha, \gamma\beta \rangle \\ 0 & 0 \end{pmatrix}$.

The composition of the functors $\psi F_J F_N : \xi(R(\Lambda)) \to A_{\xi(R(\Lambda))}$ sends any objects with fixed projective (P_3, P_1) in some module with dimension $(0,0,1,1,0,0)$ on the k-algebra

$$\begin{array}{ccc c} \circ_1^1 & \circ_2^1 & \circ_3^1 & \\ & \downdownarrows & \downarrow & \\ & \circ_1^2 & \circ_2^2 & \circ_3^2 \end{array}$$

There are two indecomposable objects in $\xi_N(R(\Lambda))$ which we need to use, X_1 and X_2, with underlying projectives $(P_3, 0)$ and $(0, P_1)$, respectively. Let $X = X_1 \oplus X_2$, then $R_X = k \times k$, $\Delta_X = 0$ and $M_X = u_1 \oplus u_2$, where u_1 corresponds to $\lambda e_3 \mapsto \lambda\gamma\alpha \otimes e_1$ and u_2 to $\lambda e_3 \mapsto \lambda\gamma\beta \otimes e_1$.

Now let $N_X = u_2$. The composition $\psi' F'_J F'_{N_X} : \xi_X(R_X) \to A_{\xi_X(R_X)}$ returns the modules with vector dimension $(1,1)$ in the algebra

$$\circ_1 \quad \to \quad \circ_2 \,.$$

We take X' an object which is sent by $\psi' F'_J$ to the indecomposable module $(1,1)$, because then we get almost all the objects with selfextensions with dimension one. Now $R'_X = k$, $\Delta_{X'} = 0$ and $M_{X'}$ is a single loop, and the objects of $H(k,1)$ in $\xi_{X'}(R_{X'})$ are parametrized by $\left(k[x], e_x^X\right)$ where $e_x^X(q) = 1_{R_{X'}} \otimes q + M_{X'} \otimes qx$.

The composition of the functors $\tau_X \tau_{X'}$ induces the parametrizer $((P_3 \otimes k[x], P_1 \otimes k[x]), e_x)$ where

$$e_x \begin{pmatrix} \lambda e_3 \otimes q_1 \\ \lambda' e_1 \otimes q_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \lambda e_3 \otimes q_1 \\ \lambda' e_1 \otimes q_2 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \lambda\gamma\alpha \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ e_1 \otimes q_1 x \end{pmatrix} + \begin{pmatrix} 0 & \lambda\gamma\beta \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ e_1 \otimes q_1 \end{pmatrix}$$

Then the functor cokernel give us the parametrizer

$$k[x] \quad \overset{(1,0)^t}{\underset{(0,1)^t}{\rightrightarrows}} \quad k[x]^2 \quad \xrightarrow{(1,-x)} \quad k[x]$$

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Universidad Autónoma de Coahuila
Departamento de Polímeros
Facultad de Ciencias Químicas
Apartado Postal 935
Ing. José Cárdenas Valdés y
Blvd. Venustiano Carranza
Col. República Oriente
25000 Saltillo, Coahuila
México
efren_math@yahoo.com.mx

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ZHAO $F(p,q,s)$ FUNCTION SPACES AND HARMONIC MAJORANTS

E. RAMÍREZ DE ARELLANO, L. F. RESÉNDIS O., AND L. M. TOVAR S.

ABSTRACT. We obtain harmonic majorants associated to analytic functions in the weighted spaces $F(p,q,s)$ introduced by Zhao in 1996.

1. Introduction

Let $0 < r$. Define $\Delta(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$ and $\Delta(r) := \Delta(0,r)$. We denote by $\Delta = \Delta(1)$ the open unit disk in the complex plane $\mathbb{C}$ and by T its boundary. Let $\mathcal{A}$ be the class of analytic functions in Δ and let $0 < \alpha < \infty$. The α-Bloch spaces are defined by (see [Zha1])

$$\mathcal{B}^\alpha := \{f \in \mathcal{A} : [f]_{\mathcal{B}^\alpha} := \sup_{z\in\Delta}(1-|z|^2)^\alpha |f'(z)| < \infty\}.$$

For $\alpha = 1$ we obtain the well known Bloch space $\mathcal{B} = \mathcal{B}^1$. R. Aulaskari, J. Xiao, and R. Zhao introduced in 1995 the $\mathcal{Q}_p$-spaces, $0 \le p < \infty$ (see [AuXiZh]) as

$$\mathcal{Q}_p := \{f \in \mathcal{A} : \ \{f\}_{\mathcal{Q}_p} := \sup_{a\in\Delta} \iint_\Delta |f'(z)|^2 g(z,a)^p \, dx\, dy < \infty\}$$

where $g(z,a)$, defined by

$$g(z,a) = \ln\left|\frac{1-\overline{a}z}{a-z}\right|,$$

is the Green's function of the unit disk with logarithmic singularity at $a \in \Delta$.

If $0 < p < q < \infty$ we have $\mathcal{B}^p \subset \mathcal{B}^q$ and $\mathcal{Q}_p \subset \mathcal{Q}_q \subset \mathcal{B}$. Aulaskari, Xiao and Zhao proved the following in [AuXiZh].

THEOREM (1.1). *Let $f \in \mathcal{A}$ and let $p \ge 0$. Then $f \in \mathcal{Q}_p$ if and only if*

$$[f]_{\mathcal{Q}_p} := \sup_{a\in\Delta} \iint_\Delta |f'(z)|^2 (1-|\phi_a(z)|^2)^p \, dx\, dy < +\infty$$

where

$$\phi_a(z) = \frac{a-z}{1-\overline{a}z}$$

is a Möbius transformation of Δ with $|a| < 1$.

For $-\infty < p < \infty$ consider the space D_p defined by

$$D_p = \{f(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{A} : \sum_{n=1}^\infty n^{1-p}|a_n|^2 < \infty\}.$$

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It can be shown (see [Ste]) that for $-1 < p$, one has $f \in D_p$ if and only if

$$\iint_\Delta |f'(z)|^2(1-|z|^2)^p \, dx\, dy < \infty.$$

Thus for $p = 0$ we obtain the classical Dirichlet space and it follows immediately that if $p < q$ then $D_p \subset D_q$ and for $0 \le p$, one has $\mathcal{Q}_p \subset D_p$.

Consider now the *Besov-type* spaces ([Mi], [St]) for $0 < p < \infty$:

$$B^p := \{f \in \mathcal{A} : \sup_{a\in\Delta} \iint_\Delta |f'(z)|^p(1-|z|^2)^{p-2}(1-|\phi_a(z)|^2)\, dx\, dy < \infty\}.$$

For $0 < p < q < \infty$, they satisfy $B^p \subset B^q$. In [AuTo] it is shown that

$$\bigcup_{0<p\le 1} \mathcal{Q}_p \subset \bigcap_{0<q\le 2} B^q, \tag{1.2}$$

and Zhao has proved in [Zha1], Theorem 1.3, the following.

THEOREM (1.3). *Let* $0 < \alpha < \infty$, $0 < p < \infty$, $1 < s < \infty$. *Then if* $f \in \mathcal{A}$, *the following statements are equivalent:*

(a) $[f]_{\mathcal{B}^\alpha} < \infty$,

(b) $\sup_{a\in\Delta} \iint_\Delta |f'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\phi_a(z)|^2)^s \, dx\, dy < \infty$,

(c) $\sup_{a\in\Delta} \iint_\Delta |f'(z)|^p(1-|z|^2)^{\alpha p-2} g^s(z,a)\, dx\, dy < \infty$.

Thus the spaces B^p are included in $\mathcal{B}^\alpha$ for $\alpha \ge 1$. All these spaces $\mathcal{B}^\alpha$, $\mathcal{Q}_p$, D_p and B^p have been extensively studied by many authors in the recent years, because they are Banach spaces with many interesting properties.

With the aim of generalizing these spaces, Zhao in [Zha2] introduced for $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$, the spaces $F(p,q,s)$ defined as

$$F(p,q,s) = \{f \in \mathcal{A} : \{f\}_{p,q,s} = \sup_{a\in\Delta} \iint_\Delta |f'(z)|^p(1-|z|^2)^q g^s(z,a)\, dx\, dy < \infty\}.$$

All the spaces mentioned previously can be studied in the framework of the spaces $F(p,q,s)$ and this opens several lines of research. Thus for instance

$$\begin{aligned} \mathcal{B}^\alpha &= F(p, \alpha p - 2, s),\ 1 < s < \infty; \\ \mathcal{Q}_s &= F(2, 0, s); \\ D_q &= F(2, q, 0),\ 0 \le q < \infty; \\ B^p &= F(p, p-2, 1). \end{aligned}$$

In his paper Zhao generalizes Theorem (1.1) for functions in $F(p,q,s)$ (see [Zha2], Theorem 2.4).

THEOREM (1.4). *Let* $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ *and let* $f \in \mathcal{A}$. *Then* $f \in F(p,q,s)$ *if and only if*

$$[f]_{p,q,s} = \sup_{a\in\Delta} \iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^s \, dx\, dy < \infty.$$

Observe that $\{f\}_{p,q,s}^{1/p}$ and $[f]_{p,q,s}^{1/p}$ define corresponding seminorms in such a way that $F(p, q, s)$ turns out to be a Banach space for $p \geq 1$, even under the norms $\{f\}_{p,q,s}^{1/p} + |f(0)|$ or $[f]_{p,q,s}^{1/p} + |f(0)|$. Rättyä in [Ra1] has obtained several important generalizations in $F(p, q, s)$ spaces. Inequality (1.2) is generalized as follows (see [Ra1] Theorem 2.2.3).

THEOREM (1.5). *Let* $0 < p^* < \infty$, $0 < \alpha < \infty$. *Then*

$$\bigcup_{0<s<1} F(p^*, \alpha p^* - 2, s) \subset \bigcap_{0<p<p^*} F(p, \alpha p - 2, 1).$$

Theorem (1.4) is also generalized for the n-th derivative of f (see [Ra1], Theorem 4.2.1, [Ra2], Theorem 3.2).

THEOREM (1.6). *Let* $f \in \mathcal{A}$ *and let* $0 < p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$ *and* $n \in \mathbb{N}$ *with* $q + s > -1$. *Then the following conditions are equivalent:*

(a) $f \in F(p, q, s)$;

(b) $\sup_{a\in\Delta} \iint_{\Delta} |f^{(n)}(z)|^p (1 - |z|^2)^{np-p+q} (1 - |\phi_a(z)|^2)^s \, dx\, dy < \infty$;

(c) $\sup_{a\in\Delta} \iint_{\Delta} |f^{(n)}(z)|^p (1 - |z|^2)^{np-p+q} g(z, a)^s \, dx\, dy < \infty$.

In [AuReTo], harmonic majorants characterizing functions in $\mathcal{Q}_p$ were introduced. These majorants constitute a new tool for studying the behaviour of functions in $\mathcal{Q}_p$ in the boundary T of Δ. The aim of this paper is to show how the main results on $F(p, q, s)$ developed up to date have corresponding ones in terms of harmonic majorants characterizing functions in $F(p, q, s)$.

In Section 2 we present the construction of the harmonic majorants associated to a continuous function in Δ, which we will use throughout this work.

In Section 3 we apply the method and the results of Section 2 for constructing a harmonic majorant characterizing a function in $F(p, q, s)$. In Section 4 we associate a harmonic majorant to every expression characterizing functions in $F(p, q, s)$ appearing in Theorems (1.3), (1.4), (1.5) and (1.6) and furthermore we see how these theorems and several important relationships among functions in $F(p, q, s)$ can be translated in terms of the corresponding harmonic majorants characterizing functions on these spaces. Finally, in Section 5 we associate to every harmonic majorant –through its corresponding conjugate function– its analytic completion. We study to which of the weighted spaces of functions quoted in the previous sections this completion belongs. This method of characterizing and studying weighted functions spaces in terms of harmonic majorants gives rise to a new line of problems.

2. Harmonic majorants and continuous functions

In this section we present several results related to continuous functions, the classical solution of the Dirichlet problem in the unit disk and harmonic majorants. Although the idea for constructing the majorants is similar to the one used in [AuReTo], in this paper we improve the method; some results and proofs are different from those presented in that paper.

Let $h : T \to \mathbb{R}$ be an integrable function on T, and $P[h]$ its Poisson extension to Δ, explicitly given by

$$P[h](z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,r}(\theta - t)h(e^{it})\,dt, \tag{2.1}$$

where $z = re^{i\theta}$, $0 \le r < 1$, $0 \le \theta \le 2\pi$, and

$$P_{R,r}(\theta) = \frac{R^2 - r^2}{R^2 - 2rR\cos\theta + r^2}$$

is the well-known Poisson kernel.

PROPOSITION (2.2). [AxBoRa] *Let $h : T \to \mathbb{R}$ be an integrable function on T. Then its Poisson extension $P[h]$ has a nontangential limit $h(e^{i\theta})$ at almost every $e^{i\theta} \in T$.*

PROPOSITION (2.3). [AuReTo] *Let $h : \Delta \to \mathbb{R}$ be a continuous function and $0 < R < 1$. Let $h_R : T \to \mathbb{R}$ be the function defined by*

$$h_R(e^{i\theta}) = \max_{0\le r\le R} h(re^{i\theta}), \qquad \theta \in [0, 2\pi). \tag{2.4}$$

Then h_R is a continuous function. Moreover, if $0 < R_1 < R_2 < 1$, then

$$h_{R_1}(e^{i\theta}) \le h_{R_2}(e^{i\theta}) \qquad \text{for every } \theta \in [0, 2\pi)\,. \tag{2.5}$$

COROLLARY (2.6). *Let $h : \Delta \to \mathbb{R}$ be a continuous function. Let $h_1 : T \to \mathbb{R}$ be defined by*

$$h_1(e^{i\theta}) = \sup_{0\le r<1} h(re^{i\theta}), \qquad \theta \in [0, 2\pi). \tag{2.7}$$

If $h_1(e^{i\theta}) < \infty$ then

$$\lim_{R\to 1^-} h_R(e^{i\theta}) = h_1(e^{i\theta}). \tag{2.8}$$

Let $h : \Delta \to \mathbb{R}$ be a continuous function and $0 < R < 1$. Let $P[h_R] : \Delta\cup T \to \mathbb{R}$ be a harmonic function on Δ such that

$$P[h_R]|T = h_R,$$

where h_R is defined by (2.4); that is, $P[h_R]$ is the classical solution of the Dirichlet problem with boundary values h_R on T. For $0 < R_1 < R_2 < 1$, by (2.5), (2.1) and the fact that $P_{1,r}(\theta) > 0$ for all θ, there holds

$$P[h_{R_1}](z) \le P[h_{R_2}](z) \le \sup_{z\in\Delta} h(z) \qquad \text{for all } z \in \Delta. \tag{2.9}$$

With the above notation, we have the following result as a consequence of Harnack's theorem.

THEOREM (2.10). *Let $h : \Delta \to \mathbb{R}$ be a continuous function. Let $h_1 : T \to \mathbb{R}$ be defined by (2.7) and let $H_1 : \Delta \to \mathbb{R}$ be defined by*

$$H_1(z) = \sup_{0\le R<1} P[h_R](z) \qquad \text{for every } z \in \Delta.$$

Then $H_1 \equiv \infty$ or H_1 is a harmonic function on Δ and

$$H_1(z) = P[h_1](z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,r}(\theta - t)h_1(e^{it})dt$$

where $z = re^{i\theta}$ and H_1 has nontangential limit

$$\lim_{w\to e^{i\theta}} H_1(w) = h_1(e^{i\theta}) \qquad \text{for a.e. } \theta \in [-\pi,\pi).$$

Proof. Suppose that H_1 is not identically ∞. Then by (2.9) and Harnack's Theorem, H_1 is a harmonic function on Δ. We consider a sequence $\{R_k\} \subset [\frac{1}{2}, 1)$ with $\lim R_k = 1$. Observe that $h_{\frac{1}{2}}$ is an integrable function. Then by (2.5) and the positivity of the Poisson kernel, $\{h_{R_k}(e^{it})P_{1,r}(\theta - t)\}$ is an increasing sequence of measurable functions with

$$\lim_{k\to\infty} h_{R_k}(e^{it})P_{1,r}(\theta - t) = h_1(e^{it})P_{1,r}(\theta - t) \qquad \text{for all } t \in [-\pi,\pi].$$

Now applying a special version of the monotone convergence theorem ([WhZy], p. 172), we obtain for all $z = re^{i\theta} \in \Delta$

$$\begin{aligned} H_1(z) &= \lim_{k\to\infty} P[h_{R_k}](z) = \lim_{k\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,r}(\theta - t)h_{R_k}(e^{it})\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{k\to\infty} P_{1,r}(\theta - t)h_{R_k}(e^{it})\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,r}(\theta - t)h_1(e^{it})\,dt. \end{aligned}$$

Also it follows that

$$H_1(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,0}(\theta - t)h_1(e^{it})\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_1(e^{it})\,dt,$$

and so h_1 is an integrable function on T. By Proposition (2.2), H_1 has a nontangential limit $h_1(e^{i\theta})$ at almost every $e^{i\theta} \in T$. □

We also obtain a converse result.

THEOREM (2.11). *Let $h : \Delta \to \mathbb{R}$ be a continuous function. Let $h_1 : T \to \mathbb{R}$ be defined by (2.7) and let $H_1 : \Delta \to \mathbb{R}$ be defined by*

$$H_1(z) = \sup_{0\le R<1} P[h_R](z) \qquad \text{for every } z \in \Delta.$$

If there exists an integrable function $g : T \to \mathbb{R}$ such that $|h_1(e^{i\theta})| \le g(e^{i\theta})$ for almost every $e^{i\theta} \in T$, then H_1 is a harmonic function on Δ and

$$H_1(z) = P[h_1](z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,r}(\theta - t)h_1(e^{it})dt$$

where $z = re^{i\theta}$ and H_1 has nontangential limit

$$\lim_{w\to e^{i\theta}} H_1(w) = h_1(e^{i\theta}) \qquad \text{for a.e. } \theta \in [-\pi,\pi).$$

Proof. Consider a sequence $\{R_k\} \subset [\frac{1}{2}, 1)$ with $\lim R_k = 1$. Observe that $h_{\frac{1}{2}}$ is an integrable function and

$$|h_{R_k}(e^{it})P_{1,r}(\theta - t)| \le \max\{|h_{\frac{1}{2}}(e^{it})|,\ g(e^{it})\}P_{1,r}(\theta - t)$$

for all $t \in [0, 2\pi)$. Applying Lebesgue's Dominated Convergence Theorem we have

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,r}(\theta - t)h_{R_k}(e^{it})\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{k\to\infty} P_{1,r}(\theta - t)h_{R_k}(e^{it})\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,r}(\theta - t)h_1(e^{it})\,dt \ < \ \infty.\end{aligned}$$

Then

$$H_1(z) = \lim_{k\to\infty} P[h_{R_k}](z) < \infty$$

and by Harnack's theorem $H_1(z)$ is a harmonic function. The existence of nontangential limits is justified as before. □

As a corollary we obtain the following.

COROLLARY (2.12). *Let $h : \Delta \to \mathbb{R}$ be a continuous function. Let $h_1 : T \to \mathbb{R}$ be defined by (2.7) and let $H_1 : \Delta \to \mathbb{R}$ be defined by*

$$H_1(z) = \sup_{0\le R<1} P[h_R](z) \qquad \text{for every } z \in \Delta.$$

Then H_1 is a harmonic function if and only if h_1 is an integrable function on T. Moreover

$$H_1(z) = P[h_1](z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,r}(\theta - t)h_1(e^{it})dt$$

where $z = re^{i\theta}$, and H_1 has nontangential limit

$$\lim_{w\to e^{i\theta}} H_1(w) = h_1(e^{i\theta}) \qquad \text{for a.e. } \theta \in \ [-\pi, \pi).$$

3. The harmonic majorant associated to a function in $F(p, q, s)$

With the aim of applying the previous construction of harmonic majorants to the expressions that characterize –after Theorem (1.4)– functions in $F(p, q, s)$ we first prove the following.

THEOREM (3.1). *Let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, $0 < r < 1$. Let $f \in \mathcal{A}$. Then*

(a) *the function ${}_g h : \Delta(r) \to \mathbb{R}$ defined by*

$${}_g h(a) = \iint_{\Delta(r)} |f'(z)|^p (1 - |z|^2)^q g^s(z, a)\,dx\,dy$$

is continuous on $\Delta(r)$;

(b) *the function ${}_\phi h : \Delta \to \mathbb{R}$ defined by*

$${}_\phi h(a) = \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q (1 - |\phi_a(z)|^2)^s\,dx\,dy$$

is continuous on Δ.

Proof. The proof consists of several steps.

(a) Let $a \in \Delta(r)$. Let $0 < \eta < \frac{r-|a|}{3}$ and define Ω as the compact set $[0, 2\pi] \times [0, \eta] \times \overline{\Delta}(a, \eta)$. Then the following lemma is immediate.

LEMMA (3.2). *Let $f \in \mathcal{A}$, $a \in \Delta$ be fixed, $0 < p < \infty$, and $0 < s \le 1$. Define $t : \Omega \to \mathbb{R}$ by*

$$t(\theta, \rho, b) = |f'(b + \rho e^{i\theta})|^p (1 - |b + \rho e^{i\theta}|^2)^q \rho \log^s \left| \frac{1 - |b|^2 - \overline{b}\rho e^{i\theta}}{\rho} \right|$$

for $\rho > 0$ and $t(\theta, 0, b) = 0$. Then t is uniformly continuous on Ω.

PROPOSITION (3.3). *Let a, η be as above and $0 < \rho_0 \le \eta$. Given $\epsilon > 0$ there exists $\delta > 0$ such that if $b, c \in \Delta(a, \eta)$ with $|b - c| < \delta$ then*

$$\left| \iint_{\Delta(b,\rho_0)} |f'(z)|^p (1 - |z|^2)^q g^s(z, b)\, dx\, dy - \iint_{\Delta(c,\rho_0)} |f'(z)|^p (1 - |z|^2)^q g^s(z, c)\, dx\, dy \right| < \epsilon.$$

Proof. We note that

$$\iint_{\Delta(d,\rho_0)} |f'(z)|^p (1 - |z|^2)^q g^s(z, d)\, dx\, dy = \int_0^{\rho_0} \int_0^{2\pi} t(\theta, \rho, d)\, d\theta d\rho.$$

By Lemma (3.2), given $\dfrac{\epsilon}{2\pi\eta} > 0$ there exists $\delta > 0$ such that $|(\theta, \rho, b) - (\theta', \rho', c)| < \delta$ implies

$$|t(\theta, \rho, b) - t(\theta', \rho', c)| < \frac{\epsilon}{2\pi\eta}.$$

Therefore

$$\int_0^{\rho_0} \int_0^{2\pi} |t(\theta, \rho, b) - t(\theta, \rho, c)|\, d\theta d\rho < \epsilon. \qquad \square$$

Let $a \in \Delta(r)$ be fixed. We know that if $f \in F(p, q, s)$ then

$$\iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q g^s(z, a)\, dx\, dy < +\infty.$$

By the absolute continuity of the integral, given $\epsilon > 0$ there exists $\eta > 0$ as before such that

$$\iint_{\Delta(a,\eta)} |f'(z)|^p (1 - |z|^2)^q g^s(z, a)\, dx\, dy < \frac{\epsilon}{6}. \tag{3.4}$$

Without loss of generality, by Proposition (3.3) there exists $0 < \delta' < \dfrac{\eta}{2}$ such that $|b - a| < \delta'$ implies

(3.5)
$$\left| \iint_{\Delta(b,\eta)} |f'(z)|^p (1 - |z|^2)^q g^s(z, b)\, dx\, dy - \iint_{\Delta(a,\eta)} |f'(z)|^p (1 - |z|^2)^q g^s(z, a)\, dx\, dy \right| < \frac{\epsilon}{6}.$$

We consider now the function defined on $D = \left[\overline{\Delta}(r) - \Delta(a, \frac{\eta}{3})\right] \times \overline{\Delta}(a, \frac{\eta}{6})$ by

$$(z, b) \to |f'(z)|^p (1 - |z|^2)^q g^s(z, b).$$

This function is uniformly continuous on D, so as in Proposition (3.3) there exists $0 < \delta'' < \min\{\delta', \frac{\eta}{6}\}$ such that $|b-a| < \delta''$ implies

$$\Big| \iint_{\overline{\Delta}(r)-\Delta(a,\frac{\eta}{3})} |f'(z)|^p(1-|z|^2)^q g^s(z,b)\,dx\,dy - \iint_{\overline{\Delta}(r)-\Delta(a,\frac{\eta}{3})} |f'(z)|^p(1-|z|^2)^q g^s(z,a)\,dx\,dy \Big| < \frac{\epsilon}{3}.$$

Since $\Delta(a,\frac{\eta}{3}) \subset \Delta(a,\eta)$, by (3.4)

$$\iint_{\Delta(a,\frac{\eta}{3})} |f'(z)|^p(1-|z|^2)^q g^s(z,a)\,dx\,dy < \frac{\epsilon}{6}.$$

In a similar form, since $\Delta(a,\frac{\eta}{3}) \subset \Delta(b,\frac{2\eta}{3})$, by (3.4) and (3.5)

$$\iint_{\Delta(a,\frac{\eta}{3})} |f'(z)|^p(1-|z|^2)^q g^s(z,b)\,dx\,dy < \frac{\epsilon}{3}.$$

The continuity of ${}_g h(a)$ follows.

(b) Let $a \in \Delta$ be fixed and let $\delta > 0$ be such that $\overline{\Delta}(a,\delta) \subset \Delta$. The function $l : \overline{\Delta} \times \overline{\Delta}(a,\delta) \to \mathbb{R}$ defined by

$$(z,\xi) \mapsto \frac{(1-|\xi|^2)^s}{|1-\bar{\xi}z|^{2s}}$$

is uniformly continuous on $\overline{\Delta} \times \overline{\Delta}(a,\delta)$. Then, assuming that f is not constant, for given $\epsilon > 0$ there exists $\rho > 0$ such that if $|z'-z| < \rho$ and $|\xi'-\xi| < \rho$ then

$$|l(z',\xi') - l(z,\xi)| < \frac{\epsilon}{\iint_\Delta |f'(z)|^p(1-|z|^2)^{q+s}\,dx\,dy}.$$

Note that

$$\iint_\Delta |f'(z)|^p(1-|z|^2)^{q+s}\,dx\,dy =_\phi h(0) < \infty$$

because $f \in F(p,q,s)$. Therefore if $|a-b| < \rho$ then

$$|\,{}_\phi h(a) - {}_\phi h(b)| \le \iint_\Delta |f'(z)|^p(1-|z|^2)^{q+s}|l(z,a)-l(z,b)|\,dx\,dy < \epsilon.$$

Therefore the theorem follows. □

Following the same line used in [AuReTo], consider an analytic function $f \in F(p,q,s)$, $0 < R < 1$, and $a = re^{i\theta}$ with $0 < r < R$. We introduce the family of functions ${}_g h_R : T \to \mathbb{R}$ defined by

$${}_g h_R(e^{i\theta}) := \max_{0\le r\le R} {}_g h(re^{i\theta}) = \max_{a=re^{i\theta},\ 0\le r\le R} \iint_{\Delta(R)} |f'(z)|^p(1-|z|^2)^q g^s(z,a)\,dx\,dy.$$

Observe that the function ${}_g h_R$ is well defined, and moreover, by Theorem (3.1) each function ${}_g h_R$ is continuous. Further as $f \in F(p,q,s)$, we have for $0 < R_1 < R_2 < 1$

$${}_g h_{R_1}(e^{i\theta}) \le {}_g h_{R_2}(e^{i\theta}) \le {}_g h_1(e^{i\theta}) \le \sup_{\theta\in[0,2\pi]} {}_g h_1(e^{i\theta}) = \{f\}_{p,q,s}.$$

Let ${}_g H_R$ be the corresponding harmonic function associated to ${}_g h_R$, that is, the solution of the Dirichlet Problem with boundary values given by ${}_g h_R$. It follows from Proposition (2.3) and Corollary (2.6) that if ${}_gH_1$ is the corresponding function defined by Theorem (2.10) then

$$\sup_{z\in\Delta} {}_gH_R(z) \le \sup_{z\in\Delta} {}_gH_1(z) = \{f\}_{p,q,s}.$$

and we are in position to give an analogue of Theorem 3.1 in [AuReTo] for characterizing functions in $F(p,q,s)$.

THEOREM (3.6). *Let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ and let $f \in \mathcal{A}$. Then f belongs to the class $F(p,q,s)$ if and only if the function ${}_gh_1 : T \to \mathbb{R}$ defined by*

$${}_gh_1(e^{i\theta}) = \sup_{a=re^{i\theta},\ 0\le r<1} \iint_\Delta |f'(z)|^p(1-|z|^2)^q g^s(z,a)\,dx\,dy$$

is bounded. The function ${}_gH_1$ is then the Dirichlet solution with boundary values given by ${}_gh_1$. Further ${}_gH_1$ is the least harmonic majorant of the family of harmonic functions $\{{}_gH_R\}$, $0 < R < 1$.

In the same way as for ${}_gh$, by Theorem (3.1) b) ${}_\phi h$ is continuous in Δ, so we introduce also the family of functions ${}_\phi h_R : T \to \mathbb{R}$ defined by

$${}_\phi h_R(e^{i\theta}) := \max_{0\le r\le R} {}_\phi h(re^{i\theta}) = \max_{a=re^{i\theta},\ 0\le r\le R} \iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^s\,dx\,dy.$$

As before, by Theorem (3.1) the functions ${}_\phi h_R$ are well defined, and induce corresponding Dirichlet solutions ${}_\phi H_R$ in the disk Δ, with boundary values given by ${}_\phi h_R$. If ${}_\phi h_1$ and ${}_\phi H_1$ are now the functions associated by Theorem (2.10) we have likewise

$$(3.7)\quad {}_\phi h_1(e^{i\theta}) := \sup_{0\le r\le 1} {}_\phi h(re^{i\theta}) = \sup_{a=re^{i\theta},\ 0\le r\le 1} \iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^s\,dx\,dy,$$

$$(3.8)\quad {}_\phi H_1(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} {}_\phi h_1(e^{i\theta})P_{1,r}(\theta-t)\,dt,$$

and

$$\sup_{z\in\Delta} {}_\phi H_R(z) \le \sup_{z\in\Delta} {}_\phi H_1(z) = \sup_{e^{i\theta}\in T} {}_\phi h_1(e^{i\theta}) = [f]_{p,q,s}.$$

Thus we are in a position to give another version of Zhao's Theorem (1.4) in terms of the harmonic majorants ${}_gH_1$ and ${}_\phi H_1$ and their corresponding boundary functions ${}_gh_1$ and ${}_\phi h_1$.

THEOREM (3.9). *Let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$. Let $f \in \mathcal{A}$. Then the following conditions are equivalent:*

(a) *$f \in F(p,q,s)$;*

(b) *${}_gh_1$ is bounded and ${}_gH_1$ is the classical Dirichlet solution with boundary values given by ${}_gh_1$;*

(c) *${}_\phi h_1$ is bounded and ${}_\phi H_1$ is the classical Dirichlet solution with boundary values given by ${}_\phi h_1$.*

Observe now that if in the proof of Theorem (3.1) we replace f' by $f^{(n)}$ the same conclusion follows. Thus we obtain the following.

COROLLARY (3.10). *Let* $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, $0 < r < 1$ *and* $n \in \mathbb{N}$. *Let* $f \in \mathcal{A}$. *Then*

(a) *the function* ${}_g h^n : \Delta(r) \to \mathbb{R}$ *defined by*

$$ {}_g h^n(a) = \iint_{\Delta(r)} |f^{(n)}(z)|^p (1 - |z|^2)^q g^s(z, a)\, dx\, dy $$

is continuous on $\Delta(r)$*;*

(b) *the function* ${}_\phi h^n : \Delta \to \mathbb{R}$ *defined by*

$$ {}_\phi h^n(a) = \iint_{\Delta} |f^{(n)}(z)|^p (1 - |z|^2)^q (1 - |\phi_a(z)|^2)^s\, dx\, dy $$

is continuous on Δ.

Thus if $f \in F(p, q, s)$, the functions ${}_g h^n$ and ${}_\phi h^n$ also induce corresponding integrable functions ${}_g h_1^n$, ${}_\phi h_1^n$ through the Poisson kernel. So after the previous Corollary and Theorem (1.6) we can translate Rättyä's result in terms of harmonic majorants.

THEOREM (3.11). *Let* $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$ *be with* $q+s > -1$. *Let* $f \in \mathcal{A}$. *Then the following conditions are equivalent:*

(a) $f \in F(p, q, s)$*;*

(b) ${}_g h_1^n$ *is bounded and* ${}_g H_1^n$ *is the classical Dirichlet solution with boundary values given by* ${}_g h_1^n$*;*

(c) ${}_\phi h_1^n$ *is bounded and* ${}_\phi H_1^n$ *is the classical Dirichlet solution with boundary values given by* ${}_\phi h_1^n$.

4. Inclusions among $F(p, q, s)$ spaces and harmonic majorants

We will now see how several important inclusions among $F(p, q, s)$ spaces can be translated in terms of the harmonic majorants characterizing functions on these spaces.

Consider the parameters p, q, s. If we fix two of them and we let the third vary we obtain

(a) For $0 < p < \infty$, $-2 < q < \infty$, $0 < s < s^* < s' \le 1$, with $q + s > -1$, as $(1 - |\phi_a(z)|^2)^{s^*} \le (1 - |\phi_a(z)|^2)^s$ we have $F(p, q, s) \subset F(p, q, s^*)$. Furthermore we have

$$ \bigcup_{0<s\le s^*<1} F(p, q, s) \subset F(p, q, s^*) \subset \bigcap_{s^*<s'\le 1} F(p, q, s'). \tag{4.1} $$

(b) For $0 < p < \infty$, $0 < s < s^* \le 1$, $-2 < q < q^* < q' < \infty$, $q + s > -1$. As $(1 - |z|^2)^{q^*} \le (1 - |z|^2)^q$ we have $F(p, q, s) \subset F(p, q^*, s)$. Furthermore we have

$$ \bigcup_{-2<q\le q^*<\infty} F(p, q, s) \subset F(p, q^*, s) \subset \bigcap_{q^*<q'<\infty} F(p, q', s). \tag{4.2} $$

(c) For $0 < p < p^* < p' < \infty$, $0 < s < s^* \le 1$, $-2 < q < \infty$ with $q + s > -1$. As

$$ \iint_{\Delta} (1 - |z|^2)^q (1 - |\phi_a(z)|^2)^s\, dx\, dy < C \quad \text{for all } a \in \Delta $$

by Hölder's inequality (see [Zha2], Proposition 6.2), we obtain

$$\iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^s\,dx\,dy < C\iint_\Delta |f'(z)|^{p^*}(1-|z|^2)^q(1-|\phi_a(z)|^2)^s\,dx\,dy,$$

so $F(p^*,q,s)\subset F(p,q,s)$. Furthermore we have

$$\bigcup_{0<p^*\le p'<\infty} F(p',q,s)\subset F(p^*,q,s)\subset \bigcap_{0<p<p^*<\infty} F(p,q,s). \tag{4.3}$$

It follows from the condition in (a) that for every $a\in\Delta$,

$$\iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^{s^*}\,dx\,dy \le \iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^s\,dx\,dy.$$

Thus if $0<R<1$ and ${}_\phi h_{R,s}$ and ${}_\phi h_{R,s^*}$ are the functions given in (2.4) associated to a function in $F(p,q,s)$ and $F(p,q,s^*)$ respectively and defined for $\theta\in[0,2\pi]$ by

$${}_\phi h_{R,s}(e^{i\theta}) = \max_{a=re^{i\theta},\ 0\le r\le R}\iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^s\,dx\,dy$$

and

$${}_\phi h_{R,s^*}(e^{i\theta}) = \max_{a=re^{i\theta},\ 0\le r\le R}\iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^{s^*}\,dx\,dy,$$

then we have ${}_\phi h_{R,s^*}(e^{i\theta})\le {}_\phi h_{R,s}(e^{i\theta})$ and their limit functions ${}_\phi h_{1,s}$ and ${}_\phi h_{1,s^*}$ (see Theorem 2.10) also satisfy ${}_\phi h_{1,s^*}(e^{i\theta})\le_\phi h_{1,s}(e^{i\theta})$ for all $\theta\in[0,2\pi]$. If we now consider their corresponding Dirichlet solutions ${}_\phi H_{1,s}$ and ${}_\phi H_{1,s^*}$, for every $z=re^{i\theta}\in\Delta$ we have

$${}_\phi H_{1,s^*}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} {}_\phi h_{1,s^*}(e^{i\theta})P_{1,r}(\theta-t)\,dt \le \frac{1}{2\pi}\int_{-\pi}^{\pi} {}_\phi h_{1,s}(e^{i\theta})P_{1,r}(\theta-t)\,dt =_\phi H_{1,s}(z).$$

Thus we arrive at the next proposition.

PROPOSITION (4.4). *Let $0<p<\infty$, $-2<q<\infty$, $0\le s<s^*<1$, with $q+s>-1$, and let $f\in\mathcal{A}$ be a function in $F(p,q,s)$. Moreover let ${}_\phi H_{1,s}$ and ${}_\phi H_{1,s^*}$ be the Dirichlet solutions with boundary values given by ${}_\phi h_{1,s}$ and ${}_\phi h_{1,s^*}$ respectively, associated to f (see (3.7) and (3.8)). Then for every $z\in\Delta$ and $\theta\in[0,2\pi]$,*

$${}_\phi h_{1,s^*}(e^{i\theta})\le {}_\phi h_{1,s}(e^{i\theta}) \quad\text{and}\quad {}_\phi H_{1,s^*}(z)\le {}_\phi H_{1,s}(z). \tag{4.5}$$

It follows from Theorem (3.9) that if ${}_\phi h_{1,s}$ is bounded on T, the inequalities (4.5) imply the same conclusion in (a).

If we apply the same kind of arguments to conditions (b) and (c), and change appropriately the corresponding indices, we obtain:

PROPOSITION (4.6). *Let $0<p<\infty$, $-2<q<q^*<\infty$, $0\le s<1$, with $q+s>-1$, and let $f\in\mathcal{A}$ be a function in $F(p,q,s)$. Moreover, let ${}_\phi H_{1,q}$ and ${}_\phi H_{1,q^*}$ be the Dirichlet solutions with boundary values given by ${}_\phi h_{1,q}$ and ${}_\phi h_{1,q^*}$ respectively, associated to f (see (3.7) and (3.8)). Then for every $z\in\Delta$ and $\theta\in[0,2\pi]$,*

$${}_\phi h_{1,q^*}(e^{i\theta})\le {}_\phi h_{1,q}(e^{i\theta}) \quad\text{and}\quad {}_\phi H_{1,q^*}(z)\le {}_\phi H_{1,q}(z).$$

PROPOSITION (4.7). *Let $0 < p < p^* < \infty$, $-2 < q < \infty$, $0 \le s < 1$, with $q + s > -1$, and let $f \in \mathcal{A}$ be a function in $F(p^*, q, s)$. Moreover, let ${}_\phi H_{1,p}$ and ${}_\phi H_{1,p^*}$ be the Dirichlet solutions with boundary values given by ${}_\phi h_{1,p}$ and ${}_\phi h_{1,p^*}$ respectively, associated to f (see (3.7) and (3.8)). Then for every $z \in \Delta$ and $\theta \in [0, 2\pi]$,*

$$ {}_\phi h_{1,p}(e^{i\theta}) \le C\, {}_\phi h_{1,p^*}(e^{i\theta}) \quad \text{and} \quad {}_\phi H_{1,p}(z) \le C\, {}_\phi H_{1,p^*}(z). $$

Now we translate Theorem (1.5) into the language of harmonic majorants.

Let $0 < p < \infty$, $0 < \alpha < \infty$ and $0 < s \le 1$. As we know from Theorem (3.1), the function

$$ {}_g h_{p,\alpha p-2,s}(a) = \iint_{\Delta(r)} |f'(z)|^p (1 - |z|^2)^{\alpha p - 2} g^s(z, a)\, dx\, dy $$

is continuous on $\Delta(r)$ for each $0 < r < 1$. So if $f \in F(p, \alpha p - 2, s)$, the function ${}_g h_{p,\alpha p-2,s}$ induces by Theorem (2.11) a pair ${}_g h_{1,p,\alpha p-2,s}$, ${}_g H_{1,p,\alpha p-2,s}$, where ${}_g H_{1,p,\alpha p-2,s}$ is a harmonic majorant characterizing f, with boundary values given by ${}_g h_{1,p,\alpha p-2,s}$ and satisfying

$$ \sup_{\theta \in [0,2\pi]} {}_g h_{1,p,\alpha p-2,s}(e^{i\theta}) = \sup_{z \in \Delta} {}_g H_{1,p,\alpha p-2,s}(z) = \{f\}_{p,\alpha p-2,s}. $$

Consider now Theorem (1.5). In particular it follows from the proof of this theorem (see Theorem 2.2.3 in [Ra1]) that for every $a \in \Delta$, $0 < p < p^* < \infty$, $0 < s < 1$,

$$ \iint_\Delta |f'(z)|^p (1 - |z|^2)^{\alpha p - 2} g(z, a)\, dx\, dy \le C \left(\iint_\Delta |f'(z)|^{p^*} (1 - |z|^2)^{\alpha p^* - 2} g^s(z, a)\, dx\, dy \right)^{p/p*}. $$

We deduce from this the inequality

$$ {}_g h_{1,p,\alpha p-2,s}(e^{i\theta}) \le C \big({}_g h_{1,p^*,\alpha p^*-2,s}(e^{i\theta}) \big)^{p/p^*}. $$

Consider now the well known inequality for $r > 1$ and $\alpha,\ \beta > 0$

$$ \alpha\beta \le \alpha^r + \beta^{r'} \quad \text{where} \quad r' = \frac{r}{r-1}. \tag{4.8} $$

If we take now the Dirichlet solutions ${}_g H_{1,p,\alpha p-2,s}$ and ${}_g H_{1,p^*,\alpha p^*-2,s}$ associated to ${}_g h_{1,p,\alpha p-2,s}$ and ${}_g h_{1,p^*,\alpha p^*-2,s}$ respectively and if $z = re^{i\theta}$, then

$$ \begin{aligned} {}_g H_{1,p,\alpha p-2,s}(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} {}_g h_{1,p,\alpha p-2,s}(e^{i\theta}) P_{1,r}(\theta - t)\, dt \\ &\le C \frac{1}{2\pi} \int_{-\pi}^{\pi} \big({}_g h_{1,p^*,\alpha p^*-2,s} \big)^{\frac{p}{p^*}} (e^{i\theta}) P_{1,r}(\theta - t)\, dt \\ &\le C \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} {}_g h_{1,p^*,\alpha p^*-2,s}(e^{i\theta}) P_{1,r}(\theta - t)\, dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{1,r}(\theta - t)\, dt \right\} \\ &\le C \big({}_g H_{1,p^*,\alpha p^*-2,s}(z) + 1 \big) \end{aligned} $$

where we have applied (4.8) with $r = p/p^*$. Thus we arrive at the next result.

THEOREM (4.9). *Let $0 < p < p^* < \infty$, $0 < s \le 1$, $0 < \alpha < \infty$, and let $f \in F(p^*, \alpha p^* - 2, s)$. Let ${}_g H_{1,p^*,\alpha p^*-2,s}$ and ${}_g H_{1,p,\alpha p-2,s}$ be the associated* *harmonic functions through the corresponding seminorms in $F(p^*, \alpha p^* - 2, s)$ and $F(p, \alpha p - 2, s)$ respectively. Then*

$$ {}_gH_{1,p,\alpha p-2,s}(z) \leq K\left({}_gH_{1,p^*,\alpha p^*-2,s}(z)+1\right) \tag{4.10}$$

where K is a constant.

It follows from Theorem (4.9) that if ${}_gH_{1,p^*,\alpha p^*-2,s}$ is bounded, then inequality (4.10) implies Theorem (1.5). Thus we have shown that Theorem (1.5) and Theorem (4.9) are equivalent.

Now we see what Theorem (1.3) looks like in terms of harmonic majorants.

PROPOSITION (4.11). *Let $0 < p < \infty$, $0 < \alpha < \infty$, $0 < s < \infty$, and let $f \in F(p, \alpha p - 2, s)$. Then for all $a \in \Delta$ and $0 < R$ we have*

$$(1-|a|^2)^{\alpha p}|f'(z)|^p \leq \frac{(s+1)^{s+1}}{\pi s^s} K \iint_\Delta |f'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\phi_a(z)|^2)^s\,dx\,dy$$

where $K = K(\alpha, p)$ is a constant.

Proof. Let $U(a, R) = \{z \in \mathbb{C} : |\phi_a(z)| < R\}$ be a pseudohyperbolic disk with center at $a \in \Delta$ and radius $R > 0$. Then it follows from the proof of Lemma 2.9 in [Zha2] that

$$\begin{aligned}\iint_\Delta |f'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\phi_a(z)|^2)^s\,dx\,dy \\ \geq (1-R^2)^s \iint_{U(a,R)} |f'(z)|^p(1-|z|^2)^{\alpha p-2}\,dx\,dy \\ \geq \frac{\pi(1-R^2)^s C R^2}{16}|f'(a)|^p(1-|a|^2)^{\alpha p},\end{aligned}$$

where $C = C(\alpha, p)$ is a constant depending only on α and p. Thus

$$(1-|a|^2)^{\alpha p}|f'(a)|^p \leq \frac{16}{\pi C R^2(1-R^2)^s}\iint_\Delta |f'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\phi_a(z)|^2)^s\,dx\,dy$$

for all $a \in \Delta$. The function $\pi x^2(1-x^2)^s$ takes its maximum at $x = \pm\frac{1}{\sqrt{1+s}}$, and so

$$\begin{aligned}(1-|a|^2)^{\alpha p}|f'(a)|^p &\leq \frac{16}{\pi C\frac{1}{1+s}(1-\frac{1}{1+s})^s}\iint_\Delta |f'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\phi_a(z)|^2)^s dx\,dy \\ &\leq \frac{16(1+s)^{1+s}}{\pi C s^s}\iint_\Delta |f'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\phi_a(z)|^2)^s dx\,dy. \qquad \square\end{aligned}$$

COROLLARY (4.12). *Let $0 < p < \infty$, $0 < \alpha < \infty$, $0 < s < \infty$. Then*

$$\left([f]_{\mathcal{B}^\alpha}\right)^p \leq \frac{16(1+s)^{1+s}}{\pi C(\alpha, p)s^s}[f]_{p,\alpha p-2,s}.$$

As we did in Section 3 for every function in $F(p, q, s)$, we can construct a least harmonic majorant denoted by $H_{\mathcal{B}^\alpha}$, associated to a function in $\mathcal{B}^\alpha$. For the previous parameters p, α, s, let ${}_\phi H_{1,p,\alpha p-2,s}$ be the corresponding one associated to the seminorm $[f]_{p,\alpha p-2,s}$ (see Section 2). Then we have

$$\left(H_{\mathcal{B}^\alpha}(z)\right)^p \leq \frac{16(1+s)^{1+s}}{\pi C(\alpha, p)s^s}\, {}_\phi H_{1,p,\alpha p-2,s}(z).$$

PROPOSITION (4.13). *Let $0 < p < \infty$, $0 < \alpha < \infty$, $1 < s < \infty$ and let $f \in \mathcal{A}$. Then there exist positive constants C_1, C_2 and C_3 such that*

$$\begin{aligned} {}_{\phi}H_{1,p,\alpha p-2,s}(z) \le C_1\, {}_{g}H_{1,p,\alpha p-2,s}(z) \le C_2 \sup_{z\in\Delta} \left(H_{\mathcal{B}^\alpha}(z)\right)^p \\ \le C_3 \sup_{z\in\Delta}\, {}_{\phi}H_{1,p,\alpha p-2,s}(z). \end{aligned}$$

Proof. We know that for every a and z in Δ, $1 - |\phi_a(z)|^2 \le 2g(z,a)$ (see [AuXiZh]). Then

$$\begin{aligned} &\iint_\Delta |f'(z)|^p (1-|z|^2)^{\alpha p-2} (1-|\phi_a(z)|^2)^s \, dx\, dy \\ \le\ & 2^s \iint_\Delta |f'(z)|^p (1-|z|^2)^{\alpha p-2} g^s(z,a)\, dx\, dy \\ \le\ & 2^s \iint_\Delta |f'(\phi_a(w))|^p (1-|\phi_a(w)|^2)^{\alpha p-2} \log^s \frac{1}{|w|} |\phi_a'(w)|^2 \, d\sigma(w) \\ =\ & 2^s \iint_\Delta |f'(\phi_a(w))|^p (1-|\phi_a(w)|^2)^{\alpha p} \log^s \frac{1}{|w|} (1-|w|^2)^{-2}\, d\sigma(w) \\ =\ & 2^s \left([f]_{\mathcal{B}^\alpha}\right)^p \iint_\Delta \left(\log^s \frac{1}{|w|}\right) (1-|w|^2)^{-2}\, d\sigma(w) \\ =\ & C_2 \left([f]_{\mathcal{B}^\alpha}\right)^p \le C_3 [f]_{p,\alpha p-2,s}. \end{aligned}$$

The result follows from these inequalities. □

As a consequence of the previous Proposition, the next theorem is equivalent to Theorem (1.3).

THEOREM (4.14). *Let $0 < p < \infty$, $0 < \alpha < \infty$, $1 < s < \infty$ and $f \in \mathcal{A}$. Then the following conditions are equivalent.*

(a) $H_{\mathcal{B}^\alpha} < \infty$,

(b) ${}_{g}H_{1,p,\alpha p-2,s} < \infty$,

(c) ${}_{\phi}H_{1,p,\alpha p-2,s} < \infty$.

5. The analytic completion of H_1

In Section 3 we obtained the harmonic majorant H_1 that characterizes a function $f \in F(p,q,s)$ with $s > 0$. Consider the harmonic conjugate of H_1, which we will denote by $\tilde{H}_1$. Then $G = H_1 + i\tilde{H}_1$ is an analytic function in Δ. The question is, to which of the weighted spaces $F(p,q,s)$ G belongs. The "nice" question would be: If f belongs to $F(p,q,s)$ and H_1 is the corresponding harmonic majorant, then does $G = H_1 + i\tilde{H}_1$ belong to $F(p,q,s)$ also? In this section we analyse these questions and we give several results about it. Thus we show that a new line of problems arises from this new way of characterizing weighted spaces of functions in terms of harmonic majorants.

Let $u : \Delta \to \mathbb{R}$ be a harmonic function and $p > 0$. Define

$$M_p(r,u) := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, \quad 0 < r < 1.$$

The function u is said to be of class h^p if $M_p(r,u)$ is uniformly bounded for all $0 < r < 1$. We observe that if u is bounded then $u \in h^p$ for all $p > 0$. An analytic function belongs to the Hardy space H^p, $0 < p$, if and only if its real and imaginary parts belong to h^p.

THEOREM (5.1). [Zy] *Let $u : \Delta \to \mathbb{R}$ be a harmonic function.*

(a) *(M. Riesz) If $u \in h^p$ for some $p \in (1,\infty)$ then its harmonic conjugate $\tilde{u}$ is also of class h^p.*

(b) *(Kolmogorov) If $u \in h^1$, then its conjugate $\tilde{u}$ is in h^p for all $p < 1$.*

(c) *(Zygmund) If $u \in h^p$ for some $p \in (1,\infty)$ then its harmonic conjugate $\tilde{u}$ is also of class h^1.*

We obtain immediately the following result.

THEOREM (5.2). *Let $h : \Delta \to \mathbb{R}$ be a bounded continuous function and let $H_1 : \Delta \to \mathbb{R}$ be the harmonic function defined for $z = re^{i\theta}$ as*

$$H_1(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{1,r}(\theta - t) \sup_{0 \le r < 1} h(re^{it})dt. \tag{5.3}$$

Then H_1, $\tilde{H}_1$ is in h^p for all $p > 0$. In particular, for each $p > 0$, the holomorphic function $H_1 + i\tilde{H}_1$ is in H^p, where H^p is the Hardy space.

The following theorem of Zygmund gives us more information about the analytic function $H_1 + i\tilde{H}_1$.

THEOREM (5.4). [Zy] *Let $u : \Delta \to \mathbb{R}$ be a harmonic function and $\tilde{u}$ its conjugate. If $f(z) = u(z) + i\tilde{u}(z)$ and $z = re^{i\theta}$ then*

$$|f'(z)| \le \frac{2}{1-r} u^*(z) \tag{5.5}$$

where $u^*(z)$ is the Poisson integral of $|u|$.

THEOREM (5.6). *Let $h : \Delta \to \mathbb{R}$ be a bounded continuous function and let $H_1 : \Delta \to \mathbb{R}$ be defined by (5.3). Then the holomorphic function $H_1 + i\tilde{H}_1$ belongs to the Bloch space.*

COROLLARY (5.7). *Let $f : \Delta \to \mathbb{C}$ be an analytic function belonging to $\mathcal{B}^\alpha$, $\alpha \ge 1$, and let H_1 and $G = H_1 + i\tilde{H}_1$ be the harmonic majorant corresponding to the $\mathcal{B}^\alpha$ seminorm of f and the analytic completion of H_1 respectively. Then G belongs to the $\mathcal{B}^\alpha$ space also.*

Consider now the next theorem (See [La] p.150)

THEOREM (5.8). *Let $f \in \mathcal{A}$. If f is univalent (or multivalent) in Δ then $f \in \mathcal{Q}_p$ if and only if $f \in \mathcal{B}$ for all $0 < p < 1$.*

It follows immediately that

COROLLARY (5.9). *Let $f \in \mathcal{A}$ be a function in $\mathcal{Q}_p$ for some $p > 0$. Let H_1 and $G = H_1 + i\tilde{H}_1$ be its harmonic majorant and the analytic completion of H_1 respectively. If G is univalent (or multivalent) then G is in $\mathcal{Q}_p$.*

We need the next lemma.

LEMMA (5.10) ([Zh], Chapter 4). *Let* $t > -1$, $c \in \mathbb{R}$ *and define* $I_{t,c} : \Delta \to \mathbb{R}$ *by*

$$I_{t,c}(a) = \iint_\Delta \frac{(1-|z|^2)^t}{|1-\overline{a}z|^{2+t+c}}\, dxdy.$$

Then

(a) *If* $c < 0$, *then* $I_{t,c}(a)$ *is bounded in* a.

(b) *If* $c > 0$, *then* $I_{t,c}(a) \sim \dfrac{1}{(1-|a|^2)^c} \qquad (|\mathrm{a}| \to 1-)$.

(c) *If* $c = 0$, *then* $I_{t,c}(a) \sim \log \dfrac{1}{1-|a|^2} \qquad (|\mathrm{a}| \to 1-)$.

Suppose now that $f \in F(p,q,s)$ for $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, and let H_1 be its associated harmonic majorant (recall $H_1 = |H_1|$) and let $G_1 = H_1 + i\tilde{H}_1$ be the analytic completion of H_1. By (5.5) we have the estimate

$$|G'(z)|^p \le \frac{4^p}{(1-|z|^2)^p} H_1^p(z) \le \frac{4^p}{(1-|z|^2)^p} [f]^p_{p,q,s}. \tag{5.11}$$

Then

$$\begin{aligned} &\iint_\Delta |G'(z)|^p (1-|z|^2)^q (1-|\phi_a(z)|^2)^s dx\, dy \\ &\qquad \le 4^p [f]^p_{p,q,s} \iint_\Delta \frac{(1-|z|^2)^q}{(1-|z|^2)^p}(1-|\phi_a(z)|^2)^s dx\, dy \\ &\qquad = 4^p [f]^p_{p,q,s} (1-|a|^2)^s \iint_\Delta \frac{(1-|z|^2)^{q-p+s}}{|1-\overline{a}z|^{2s}} dx\, dy\,. \end{aligned} \tag{5.12}$$

If $q - p + s > -1$, by Lemma (5.10) it follows that if

$$I(p,q,s)(a) = \iint_\Delta \frac{(1-|z|^2)^{q-p+s}}{|1-\overline{a}z|^{2s}} dx\, dy$$

then

(a) $I(p,q,s)(a)$ is bounded if $s + p - q - 2 < 0$,

(b) $I(p,q,s)(a) \sim \dfrac{1}{(1-|a|^2)^{s+p-q-2}}$ if $s + p - q - 2 > 0 \qquad (|\mathrm{a}| \to 1-)$,

(c) $I(p,q,s)(a) \sim \log \dfrac{1}{(1-|a|^2)}$ if $s + p - q - 2 = 0 \qquad (|\mathrm{a}| \to 1-)$.

Thus if $s + p - q - 2 \le 0$ we have that the right side of (5.12) is bounded in a, $G \in F(p,q,s)$ and $[G]_{p,q,s} \le C[f]^p_{p,q,s}$. If $s + p - q - 2 > 0$, to bound the right side in (5.12) it is sufficient that $p - q \le 2$.

If $0 < s \le 1$ and $q + s - p > -1$ it follows that $p - q \le 1 + s \le 2$. If $s > 1$ and $p - q \le 2$, then $-1 \le s - 2 \le q + s - p$. We arrive at the next result.

THEOREM (5.13). *Let* $f \in \mathcal{A}$. *Suppose that* f *belongs to* $F(p,q,s)$ *for some* $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$. *Let* H_1 *and* $G = H_1 + i\tilde{H}_1$ *be its harmonic majorant and the analytic completion of* H_1, *respectively. Then* G *belongs to* $F(p,q,s)$ *if*

(a) $0 < s \le 1$ *and* $q + s - p > -1$,

(b) $s > 1$ *and* $p - q \le 2$.

In both cases $[G]_{p,q,s} \le C[f]^p_{p,q,s}$.

Open Problem. In general let $f \in F(p, q, s)$ and H_1 be the corresponding harmonic majorant characterizing f. If $G = H_1 + i\tilde{H}_1$ then does G belong to $F(p, q, s)$ also?

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E. RAMÍREZ DE ARELLANO
DEPARTAMENTO DE MATEMÁTICAS, CINVESTAV IPN
AV. INSTITUTO POLITÉCNICO NACIONAL 2508
07360 MÉXICO, D.F. MÉXICO
eramirez@math.cinvestav.mx

LINO F. RESÉNDIS O.
AREA DE ANÁLISIS MATEMÁTICO Y SUS APLICACIONES
UNIVERSIDAD AUTÓNOMA METROPOLITANA, UNIDAD AZCAPOTZALCO
C.B.I.
APARTADO POSTAL 16-306
02200 MÉXICO, D.F. MÉXICO
lfro@correo.azc.uam.mx

LUIS M. TOVAR S.
ESCUELA SUPERIOR DE FÍSICA Y MATEMÁTICAS DEL IPN
EDIF. 9, UNIDAD ALM, ZACATENCO,
07300 MÉXICO, D.F., MÉXICO.
tovar@esfm.ipn.mx

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ON THE GENERAL QUADRATIC FUNCTIONAL EQUATION

JOHN MICHAEL RASSIAS

ABSTRACT. In 1940 and in 1964 S. M. Ulam proposed the general problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?". In 1941 D. H. Hyers solved this stability problem for linear mappings. In 1951 D. G. Bourgin was the second author to treat the same problem for additive mappings. According to P. M. Gruber (1978) this kind of stability problems are of particular interest in probability theory and in the case of functional equations of different types. In 1981 F. Skof was the first author to solve the Ulam problem for quadratic mappings. In 1982–2002 we solved the above Ulam problem for linear and nonlinear mappings and established analogous stability problems even on restricted domains. Further, we applied some of our recent results to the asymptotic behavior of functional equations of different types. The purpose of this paper is the stability result for generalized quadratic mappings.

1. Introduction

In 1940 and in 1964 S. M. Ulam [27] proposed the general problem:

"When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?".

In 1941 D. H. Hyers [13] solved this stability problem for linear mappings. In 1951 D. G. Bourgin [3] was the second author to treat the same problem for additive mappings. According to P. M. Gruber [12] (1978), this kind of stability problems are of particular interest in *probability theory* and in the case of *functional equations* of different types. In 1978 Th. M. Rassias [22] employed Hyers' ideas to new additive mappings. In 1981 and 1983 F. Skof [23], [24] was the first author to solve the Ulam problem for quadratic mappings. In 1982-2002 we ([16], [17], [18], [19], [20], [21]) solved the above Ulam problem for linear and nonlinear mappings and established analogous stability problems on restricted domains (see also [14]). Further, we applied some of our recent results to the asymptotic behavior of functional equations of different types. In 1999 P. Gavruta [11] answered a question of ours [16] concerning the stability of Cauchy equation. In 1996 and 1998 we [19], [20] solved the Ulam stability problem for quadratic mappings $Q : X \to Y$ satisfying the functional equation

$$Q(a_1x_1 + a_2x_2) + Q(a_2x_1 - a_1x_2) = (a_1^2 + a_2^2)[Q(x_1) + Q(x_2)]$$

for every $x_1, x_2 \in X$, and fixed reals $a_1, a_2 \neq 0$, where X and Y are real normed linear spaces. The purpose of this paper is the stability result for generalized

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quadratic mappings $Q : X \to Y$ satisfying $Q(0) = 0$ and the following quadratic functional equation

$$(*) \qquad Q\left(\sum_{i=1}^{p} a_i x_i\right) + \sum_{1\le i<j\le p} Q(a_j x_i - a_i x_j) = m \sum_{i=1}^{p} Q(x_i)$$

for every $x_i \in X (i = 1, 2, \ldots, p)$, and fixed $a_i \neq 0$ $(i = 1, 2, \ldots, p)$, $a_i \in \mathbb{R}$ $(i = 1, 2, \ldots, p)$, where $\mathbb{R} :=$ set of reals and p is arbitrary but fixed and equals to $2, 3, 4, \ldots,$ such that $0 < m = \sum_{i=1}^{p} a_i^2$.

If X and Y are normed linear spaces and Y is complete, then we establish an approximation of approximately quadratic mappings $f : X \to Y$ by quadratic mappings $Q : X \to Y$, such that $f(0) = 0$ and the corresponding approximately quadratic functional inequality

$$(**) \quad \left\| f\left(\sum_{i=1}^{p} a_i x_i\right) + \sum_{1\le i<j\le p} f(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(x_i)\right] \right\| \le c \prod_{i=1}^{p} ||x_i||^{r_i}$$

holds with constants $c \geq 0$ (independent of $x_i \in X : i = 1, 2, \ldots, p$), and any fixed reals a_i and $r_i > 0 (i = 1, 2, \ldots, p)$. Denote

$$I_1 = \{(r, m) \in \mathbb{R}^2 : 0 < r < 2,\ m > 1 \quad \text{or} \quad r > 2,\ 0 < m < 1\},$$
$$I_2 = \{(r, m) \in \mathbb{R}^2 : 0 < r < 2,\ 0 < m < 1 \quad \text{or} \quad r > 2,\ m > 1\},$$
$$I_3 = \{(r, m) \in \mathbb{R}^2 : 0 < r < 2,\ m = 1 = pb^2,\ a_i = b = p^{-1/2} : i = 1, 2, \ldots, p\},$$
$$I_4 = \{(r, m) \in \mathbb{R}^2 : r > 2,\ m = 1 = pb^2,\ a_i = b = p^{-1/2} : i = 2, \ldots, p\},$$

where $r = \sum_{i=1}^{p} r_i > 0$, where p is arbitrary but fixed and equals to $2, 3, 4, \ldots$. Note that $m^{r-2} < 1$ if $(r, m) \in I_1$, $m^{2-r} < 1$ if $(r, m) \in I_2$, $p^{r-2} < 1$ if $(r, m) \in I_3$, and $p^{2-r} < 1$ if $(r, m) \in I_4$. Also denote $\gamma = \prod_{i=1}^{p} |a_i|^{r_i} > 0$. Also denote

$$f_n(x) = \begin{cases} m^{-2n} f(m^n x), & \text{if} \quad (r, m) \in I_1 \\ m^{2n} f(m^{-n} x), & \text{if} \quad (r, m) \in I_2 \\ p^{-n} f(p^{n/2} x), & \text{if} \quad (r, m) \in I_3 \\ p^{n} f(p^{-n/2} x), & \text{if} \quad (r, m) \in I_4 \end{cases}$$

for all $x \in X$ and $n \in \mathbb{N} : p = 2, 3, 4, \ldots$.

Definition (1.1). Let X and Y be real normed linear spaces. Let $a = (a_1, a_2, \ldots, a_p) \neq (0, 0, \ldots, 0)$ with $a_i \in \mathbb{R}$ $(i = 1, 2, \ldots, p)$. Then a mapping $Q : X \to Y$ is called *quadratic with respect to* $a : |a| = \left(\sum_{i=1}^{p} a_i^2\right)^{1/2}$, if the generalized quadratic functional equation (*) holds for every $x_i \in X (i = 1, 2, \ldots, p)$. Denote

$$(1.2) \qquad \overline{Q}(x) = \begin{cases} \sum_{i=1}^{p} Q(a_i x) / \sum_{i=1}^{p} a_i^2, & \text{if} \quad (r, m) \in I_1, \\ (\sum_{i=1}^{p} a_i^2)\left[\sum_{i=1}^{p} Q\left(a_i x / \sum_{i=1}^{p} a_i^2\right)\right], & \text{if} \quad (r, m) \in I_2, \end{cases}$$

for all $x \in X$.

2. Quadratic functional stability

THEOREM (2.1). *Let X and Y be normed linear spaces. Assume that Y is complete. Assume in addition that the mapping $f : X \to Y$ satisfies $f(0) = 0$ and the approximately quadratic functional inequality (**) for every $x_i \in X$ ($i = 1, 2, \ldots, p$). If $r \neq 2$ and $p \geq 2$, then the limit*

$$Q(x) = \lim_{n\to\infty} f_n(x) \tag{2.2}$$

exists for all $x \in X$ and $Q : X \to Y$ is the unique quadratic mapping such that

$$||f(x) - Q(x)|| \leq ||x||^r \begin{cases} \gamma c/(m^2 - m^r), & \text{if} \quad (r, m) \in I_1 \\ \gamma c/(m^r - m^2), & \text{if} \quad (r, m) \in I_2 \\ c/(p - p^{r/2}), & \text{if} \quad (r, m) \in I_3 \\ c/(p^{r/2} - p), & \text{if} \quad (r, m) \in I_4 \end{cases} \tag{2.3}$$

holds for all $x \in X$.

Proof. From the hypotheses of this theorem, the following condition

$$f(0) = 0 \tag{2.4}$$

is useful to hold. We claim for each $n \in \mathbb{N}$ that

$$||f(x) - f_n(x)|| \leq ||x||^r \begin{cases} \frac{\gamma c}{m^2 - m^r}(1 - m^{n(r-2)}), & \text{if} \quad (r, m) \in I_1, \\ \frac{\gamma c}{m^r - m^2}(1 - m^{n(2-r)}), & \text{if} \quad (r, m) \in I_2, \\ \frac{c}{p - p^{r/2}}(1 - p^{n(r-2)/2}), & \text{if} \quad (r, m) \in I_3, \\ \frac{c}{p^{r/2} - p}(1 - p^{n(2-r)/2}), & \text{if} \quad (r, m) \in I_4 \end{cases} \tag{2.5}$$

for all $x \in X$. By replacing $Q, \overline{Q}$ of (1.2) with $f, \overline{f}$, respectively, one denotes:

$$\overline{f}(x) = \begin{cases} \sum_{i=1}^{p} f(a_i x) / \sum_{i=1}^{p} a_i^2, & \text{if} \quad (r, m) \in I_1 \\ \\ (\sum_{i=1}^{p} a_i^2) \left[\sum_{i=1}^{p} f\left(a_i x / \sum_{i=1}^{p} a_i^2 \right) \right], & \text{if} \quad (r, m) \in I_2 \end{cases} \tag{2.6}$$

holds for all $x \in X$. From (2.4), (2.6) and (**), with $x_i = a_i x$ ($i = 1, 2, \ldots, p$), we obtain

$$\left\| f(mx) + \binom{p}{2} f(0) - m \sum_{i=1}^{p} f(a_i x) \right\| \leq \gamma c \|x\|^r, \quad \text{or}$$

$$\left|\left| f(mx) - m \left[\sum_{i=1}^{p} f(a_i x) \right] \right|\right| \leq \gamma c ||x||^r, \quad \text{or}$$

$$||m^{-2} f(mx) - \overline{f}(x)|| \leq \frac{\gamma c}{m^2} ||x||^r, \tag{2.7}$$

if I_1 holds. Besides from (2.4), (2.6) and (**), with $x_1 = x, x_j = 0$ ($j = 2, 3, \ldots, p$), we get

$$\left|\left| f(a_1 x) + \sum_{j=2}^{p} f(a_j x) - m[f(x) + (p-1) f(0)] \right|\right| \leq 0,$$

$$\text{or}\quad \left\|\sum_{i=1}^{p} f(a_i x) - mf(x)\right\| \le 0, \quad \text{or}$$

$$\overline{f}(x) = f(x), \tag{2.8}$$

if I_1 holds. Therefore from (2) and (2.8) we have

$$||f(x) - m^{-2}f(mx)|| \le \frac{\gamma c}{m^2}||x||^r = \frac{\gamma c}{m^2 - m^r}(1 - m^{r-2})||x||^r, \tag{2.9}$$

which is (2.5) for $n = 1$, if I_1 holds.

Similarly, from (2.4), (2.6) and (**), with $x_i = \frac{a_i}{m}x (i = 1, 2, \ldots, p)$, we obtain

$$\left\|f(x) + \binom{p}{2}f(0) - m\sum_{i=1}^{p} f\left(\frac{a_i}{m}x\right)\right\| \le \frac{\gamma c}{m^r}||x||^r, \quad \text{or}$$

$$||f(x) - \overline{f}(x)|| \le \frac{\gamma c}{m^r}||x||^r, \tag{2.10}$$

if I_2 holds. Further from (2.4), (2.6) and (**), with $x_1 = \frac{x}{m}, x_j = 0$ $(j = 2, 3, \ldots, p)$, we get

$$\left\|f\left(\frac{a_1}{m}x\right) + \sum_{j=2}^{p} f\left(\frac{a_j}{m}x\right) - m[f(m^{-1}x) + (p-1)f(0)]\right\| \le 0, \quad \text{or}$$

$$\left\|\sum_{i=1}^{p} f\left(\frac{a_i}{m}x\right) - mf(m^{-1}x)\right\| \le 0, \quad \text{or}$$

$$\overline{f}(x) = m^2 f(m^{-1}x), \tag{2.11}$$

if I_2 holds. Therefore from (2.10) and (2.11) we have

$$||f(x) - m^2 f(m^{-1}x)|| \le \frac{\gamma c}{m^r}||x||^r = \frac{\gamma c}{m^r - m^2}(1 - m^{2-r})||x||^r, \tag{2.12}$$

which is (2.5) for $n = 1$, if I_2 holds.

Also, with $x_i = x$ $(i = 1, 2, \ldots, p)$ in (**) and $a_i = b = p^{-1/2}$ $(i = 1, 2, \ldots, p)$, we obtain

$$\left\|f(p^{1/2}x) - pf(x)\right\| \le c||x||^r, \quad \text{or}$$

$$\left\|f(x) - p^{-1}f(p^{1/2}x)\right\| \le \frac{c}{p}\|x\|^r = \frac{c}{p - p^{r/2}}\left[1 - p^{(r-2)/2}\right]\|x\|^r, \tag{2.13}$$

which is (2.5) for $n = 1$, if I_3 holds.

In addition, with $x_i = p^{-1/2}x$ $(i = 1, 2, \ldots, p)$ in (**) and $a_i = b = p^{-1/2}$ $(i = 1, 2, \ldots, p)$, we obtain

$$\left\|f(x) - pf(p^{-1/2}x)\right\| \le cp^{-r/2}||x||^r, \quad \text{or}$$

$$\left\|f(x) - pf(p^{-1/2}x)\right\| \le cp^{-r/2}||x||^r = \tfrac{c}{p^{r/2} - p}[1 - p^{(2-r)/2}]||x||^r, \tag{2.13a}$$

which is (2.5) for $n = 1$, if I_4 holds.

Assume (2.5) is true if $(r, m) \in I_1$. From (2.9), with $m^n x$ in place of x, and from the triangle inequality, we have

$$||f(x) - f_{n+1}(x)|| = ||f(x) - m^{-2(n+1)} f(m^{n+1}x)||$$

$$\begin{aligned} &\le ||f(x) - m^{-2n} f(m^n x)|| + ||m^{-2n} f(m^n x) - m^{-2(n+1)} f(m^{n+1}x)|| \\ &\le \frac{\gamma c}{m^2 - m^r}[(1 - m^{n(r-2)}) + m^{-2n}(1 - m^{r-2})m^{nr}]||x||^r \\ &= \frac{\gamma c}{m^2 - m^r}(1 - m^{(n+1)(r-2)})||x||^r, \end{aligned} \tag{2.14}$$

if I_1 holds.

Similarly assume (2.5) is true if $(r, m) \in I_2$. From (2.12), with $m^{-n}x$ in place of x, and the triangle inequality, we have

$$||f(x) - f_{n+1}(x)|| = ||f(x) - m^{2(n+1)} f(m^{-(n+1)}x)||$$

$$\begin{aligned} &\le ||f(x) - m^{2n} f(m^{-n} x)|| + ||m^{2n} f(m^{-n} x) - m^{2(n+1)} f(m^{-(n+1)}x)|| \\ &\le \frac{\gamma c}{m^r - m^2}[(1 - m^{n(2-r)}) + m^{2n}(1 - m^{2-r})m^{-nr}]||x||^r \\ &= \frac{\gamma c}{m^r - m^2}(1 - m^{(n+1)(2-r)})||x||^r, \end{aligned} \tag{2.15}$$

if I_2 holds.

Also, assume (2.5) is true if $(r, m) \in I_3$. From (2.13), with $(pb)^n x$ $(= p^{n/2}x)$ in place of x, and the triangle inequality, we have

$$||f(x) - f_{n+1}(x)|| = \left|\left|f(x) - p^{-(n+1)} f\left(p^{\frac{n+1}{2}} x\right)\right|\right| = ||f(x) - p^{-(n+1)} f((pb)^{n+1}x)||$$

$$\begin{aligned} &\le ||f(x) - p^{-n} f((pb)^n x)|| + ||p^{-n} f((pb)^n x) - p^{-(n+1)} f((pb)^{n+1}x)|| \\ &\le \frac{c}{p - p^{r/2}}\{[1 - p^{n(r-2)/2}] + p^{-n}[1 - p^{(r-2)/2}](pb)^{nr}\}||x||^r \\ &= \frac{c}{p - p^{r/2}}[1 - p^{(n+1)(r-2)/2}]||x||^r, \end{aligned} \tag{2.16}$$

if I_3 holds.

In addition, assume (2.5) is true if $(r, m) \in I_4$. From (2.13a), with $(pb)^n x$ $(= p^{n/2}x)$ in place of x, and the triangle inequality, we have

$$||f(x) - f_{n+1}(x)|| = \left|\left|f(x) - p^{n+1} f\left(p^{\frac{n+1}{2}} x\right)\right|\right| = ||f(x) - p^{n+1} f((pb)^{-(n+1)}x)||$$

$$\begin{aligned} &\le ||f(x) - p^n f((pb)^{-n} x)|| + ||p^n f((pb)^{-n} x) - p^{n+1} f((pb)^{-(n+1)}x)|| \\ &\le \frac{c}{p^{r/2} - p}\{[1 - p^{n(2-r)/2}] + p^n[1 - p^{(2-r)/2}](pb)^{-nr}\}||x||^r \\ &= \frac{c}{p^{r/2} - p}[1 - p^{(n+1)(2-r)/2}]||x||^r, \end{aligned} \tag{2.16a}$$

if I_4 holds.

Therefore inequalities (2.14), (2.15) and (2.16) and (2.16a) prove inequality (2.5) for any $n \in \mathbb{N}$.

We claim now that the sequence $\{f_n(x)\}$ *converges*. To do this it suffices to prove that it is a Cauchy sequence. Inequality (2.5) is involved if $(r,m) \in I_1$. In fact, if $i > j > 0$ and $h_1 = m^j x$, we have:

$$||f_i(x)-f_j(x)||=||m^{-2i}f(m^ix)-m^{-2j}f(m^jx)||=m^{-2j}||m^{-2(i-j)}f(m^{i-j}h_1)-f(h_1)||$$

$$\text{(2.17)} \quad \leq \quad m^{-2j}\frac{\gamma c}{m^2-m^r}(1-m^{(i-j)(r-2)})||h_1||^r < \frac{\gamma c}{m^2-m^r}m^{-2j}||h_1||^r \underset{j\to\infty}{\longrightarrow} 0,$$

if I_1 holds: $m^{r-2} < 1$.

Similarly, if $h_2 = m^{-j}x$ in I_2, we have:

$$||f_i(x)-f_j(x)||=||m^{2i}f(m^{-i}x)-m^{2j}f(m^{-j}x)||=m^{2j}||m^{2(i-j)}f(m^{-(i-j)}h_2)-f(h_2)||$$

$$\text{(2.18)} \quad \leq \quad m^{2j}\frac{\gamma c}{m^r-m^2}(1-m^{(i-j)(2-r)})||h_2||^r < \frac{\gamma c}{m^r-m^2}m^{2j}||h_2||^r \underset{j\to\infty}{\longrightarrow} 0,$$

if I_2 holds: $m^{2-r} < 1$.

Also, if $h_3 = p^{j/2}x$ in I_3, we have:

$$\begin{aligned} \|f_i(x)-f_j(x)\| &= \|p^{-i}f(p^{i/2}x)-p^{-j}f(p^{j/2}x)\| \\ &= p^{-j}\|p^{-(i-j)}f(p^{(i-j)/2}h_3)-f(h_3)\| \end{aligned}$$

$$\text{(2.19)} \qquad \leq p^{-j}\frac{c}{p-p^{r/2}}(1-p^{(i-j)(r-2)/2})\|h_3\|^r < \frac{c}{p-p^{r/2}}p^{-j}\|h_3\|^r \underset{j\to\infty}{\longrightarrow} 0,$$

if I_3 holds: $p^{r-2} < 1$.

In addition, if $h_4 = p^{-j/2}x$ in I_4, we have:

$$\begin{aligned} ||f_i(x)-f_j(x)|| &= ||p^if(p^{-i/2}x)-p^jf(p^{-j/2}x)|| \\ &= p^j||p^{i-j}f(p^{-(i-j)/2}h_4)-f(h_4)|| \end{aligned}$$

$$\text{(2.19a)} \qquad \leq p^j\frac{c}{p^{r/2}-p}(1-p^{(i-j)(2-r)/2})||h_4||^r < \frac{c}{p^{r/2}-p}p^j||h_4||^r \underset{j\to\infty}{\longrightarrow} 0,$$

if I_4 holds: $p^{2-r} < 1$.

Then inequalities (2.17), (2.18) and (2.19) and (2.19a) *define a mapping* $Q: X \to Y$ in p variables $x_i \in X$ $(i = 1, 2, \ldots, p)$, given by (2.2).

Claim that from (**) and (2.2) we can get (*), or equivalently that the aforementioned *well–defined mapping* $Q : X \to Y$ *is quadratic with respect to a* $(\neq 0)$. In fact, it is clear from the functional inequality (**) and the limit (2.2) for $(r,m) \in I_1$ that the following functional inequality

$$\begin{aligned} m^{-2n}\Big\|f\Big(\sum_{i=1}^{p}a_im^nx_i\Big) \;+\; \sum_{1\leq i<j\leq p}f(a_jm^nx_i-a_im^nx_j)-\Big(\sum_{i=1}^{p}a_i^2\Big)\Big[\sum_{i=1}^{p}f(m^nx_i)\Big]\Big\| \\ \leq\; m^{-2n}c\prod_{i=1}^{p}||m^nx_i||^{r_i}, \end{aligned}$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, and all $n \in \mathbb{N}$ with $p = 2, 3, 4, \ldots$ and $f_n(x) = m^{-2n}f(m^nx) : I_1$ holds. Therefore

$$\Big\|\lim_{n\to\infty}f_n\Big(\sum_{i=1}^{p}a_ix_i\Big)+\lim_{n\to\infty}\sum_{1\leq i<j\leq p}f_n(a_jx_i-a_ix_j)-\Big(\sum_{i=1}^{p}a_i^2\Big)\Big[\sum_{i=1}^{p}\lim_{n\to\infty}f_n(x_i)\Big]\Big\|$$

$$\leq (\lim_{n\to\infty} m^{n(r-2)})c \prod_{i=1}^{p} ||x_i||^{r_i} = 0,$$

because $m^{r-2} < 1$ or

$$\left\| Q\left(\sum_{i=1}^{p} a_i x_i\right) + \sum_{1\leq i<j\leq p} Q(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} Q(x_i)\right]\right\| = 0, \tag{2.20}$$

i.e., the mapping Q satisfies the quadratic functional equation (*).

Similarly, from (**) and (2.2) for $(r, m) \in I_2$ we get that

$$m^{2n}\left\| f\left(\sum_{i=1}^{p} a_i m^{-n} x_i\right) + \sum_{1\leq i<j\leq p} f(a_j m^{-n} x_i - a_j m^{-n} x_j) = \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(m^{-n} x_i)\right]\right\|$$

$$\leq m^{2n} c \prod_{i=1}^{p} ||m^{-n} x_i||^{r_i},$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, and all $n \in \mathbb{N}$ with $f_n(x) = m^{2n} f(m^{-n} x)$: I_2 holds. Thus

$$\left\| \lim_{n\to\infty} f_n\left(\sum_{i=1}^{p} a_i x_i\right) + \lim_{n\to\infty} \sum_{1\leq i<j\leq p} f_n(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} \lim_{n\to\infty} f_n(x_i)\right]\right\|$$

$$\leq (\lim_{n\to\infty} m^{n(2-r)})c \prod_{i=1}^{p} ||x_i||^{r_i} = 0,$$

because $m^{2-r} < 1$, i.e., (2.20) holds and the mapping Q satisfies (*).

Also, from (**) and (2.2) for $(r, m) \in I_3$ we obtain that

$$p^{-n}\left\| f\left(\sum_{i=1}^{p} a_i p^{n/2} x_i\right) + \sum_{1\leq i<j\leq p} f(a_i p^{n/2} x_j - a_j p^{n/2} x_i) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(p^{n/2} x_i)\right]\right\|$$

$$\leq p^{-n} c \prod_{i=1}^{p} ||p^{n/2} x_i||^{r_i},$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, and all $n \in \mathbb{N}$ with $f_n(x) = p^{-n} f(p^{n/2} x)$: I_3 holds. Hence

$$\left\| \lim_{n\to\infty} f_n\left(\sum_{i=1}^{p} a_i x_i\right) + \lim_{n\to\infty} \sum_{1\leq i<j\leq p} f_n(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} \lim_{n\to\infty} f_n(x_i)\right]\right\|$$

$$\leq (\lim_{n\to\infty} p^{n(r-2)/2})c \prod_{i=1}^{p} ||x_i||^{r_i} = 0,$$

because $p^{r-2} < 1$, i.e., (2.20) holds and the mapping Q satisfies (*).

In addition, from (**) and (2.2) for $(r, m) \in I_4$ we obtain that

$$p^{n}\left\| f\left(\sum_{i=1}^{p} a_i p^{-n/2} x_i\right) + \sum_{1\leq i<j\leq p} f(a_j p^{-n/2} x_i - a_i p^{-n/2} x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(p^{-n/2} x_i)\right]\right\| \leq p^{n} c \prod_{i=1}^{p} ||p^{-n/2} x_i||^{r_i},$$

holds for all vectors $(x_1, x_2, \ldots, x_p)\in X^p$, and all $n \in \mathbb{N}$ with $f_n(x) = p^n f(p^{-n/2}x)$: I_4 holds. Hence

$$\left|\left|\lim_{n\to\infty} f_n\left(\sum_{i=1}^{p} a_i x_i\right) + \lim_{n\to\infty} \sum_{1\le i<j\le p} f_n(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} \lim_{n\to\infty} f_n(x_i)\right]\right|\right|$$

$$\le (\lim_{n\to\infty} p^{n(2-r)/2})c \prod_{i=1}^{p} ||x_i||^{r_i} = 0,$$

because $p^{2-r} < 1$, i.e., (2.20) holds and the mapping Q satisfies (*).

Therefore (2.20) holds if I_j $(j = 1, 2, 3, 4)$ hold or the mapping Q satisfies the quadratic functional equation (*), completing the proof that Q *is a quadratic mapping with respect to* a in X. It is now clear from (2.5) with $n \to \infty$, as well as from the formula (2.2) that the functional inequality (2.3) holds in X. This completes *the existence proof* of the afore-mentioned Theorem (2.1).

It remains to prove *the uniqueness: Let* $Q' : X \to Y$ *be a quadratic mapping with respect to a satisfying* (2.3)*, as well as* Q*. Then* $Q' = Q$.

In fact, the condition

$$Q(x) = \begin{cases} m^{-2n}Q(m^n x), & \text{if} \quad (r, m) \in I_1, \\ m^{2n}Q(m^{-n}x), & \text{if} \quad (r, m) \in I_2, \\ p^{-n}Q(p^{n/2}x), & \text{if} \quad (r, m) \in I_3, \\ p^n Q(p^{-n/2}x), & \text{if} \quad (r, m) \in I_4 \end{cases} \tag{2.21}$$

holds for all $x \in X$ and $n \in \mathbb{N}$ where p is arbitrary but fixed and equals $2, 3, 4, \ldots$, as a consequence of (2.5) with $c = 0$. Remember Q' satisfies (2.21) as well for $(r, m) \in I_1$. Then for every $x \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned} ||Q(x) - Q'(x)|| &= ||m^{-2n}Q(m^n x) - m^{-2n}Q'(m^n x)|| \\ &\le m^{-2n}\left\{||Q(m^n x) - f(m^n x)|| + ||Q'(m^n x) - f(m^n x)||\right\} \\ &\le m^{-2n}\frac{2\gamma c}{m^2 - m^r}||m^n x||^r = m^{n(r-2)}\frac{2\gamma c}{m^2 - m^r}||x||^r \to 0 \\ &\quad \text{as } n \to \infty, \end{aligned} \tag{2.22}$$

if I_1 holds: $m^{r-2} < 1$.

Similarly for $(r, m) \in I_2$, we establish

$$\begin{aligned} ||Q(x) - Q'(x)|| &= ||m^{2n}Q(m^{-n}x) - m^{2n}Q'(m^{-n}x)|| \\ &\le m^{2n}\left\{||Q(m^{-n}x) - f(m^{-n}x)|| + ||Q'(m^{-n}x) - f(m^{-n}x)||\right\} \\ &\le m^{2n}\frac{2\gamma c}{m^r - m^2}||m^{-n}x||^r = m^{n(2-r)}\frac{2\gamma c}{m^r - m^2}||x||^r \to 0, \\ &\quad \text{as } n \to \infty, \end{aligned} \tag{2.23}$$

if I_2 holds: $m^{2-r} < 1$.

Also for $(r, m) \in I_3$, we get

$$\begin{aligned} ||Q(x) - Q'(x)|| &= ||p^{-n}Q(p^{n/2}x) - p^{-n}Q'(p^{n/2}x)|| \\ &\leq p^{-n}\left\{||Q(p^{n/2}x) - f(p^{n/2}x)|| + ||Q'(p^{n/2}x) - f(p^{n/2}x)||\right\} \\ &\leq p^{-n}\frac{2c}{p - p^{r/2}}||p^{n/2}x||^r = p^{n(r-2)/2}\frac{2c}{p - p^{r/2}}||x||^r \to 0, \\ &\quad \text{as } n \to \infty, \end{aligned} \tag{2.24}$$

if I_3 holds: $p^{r-2} < 1$.

In addition, for $(r, m) \in I_4$, we get

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \|p^n Q(p^{-n/2}x) - p^n Q'(p^{-n/2}x)\| \\ &\leq p^n\left\{\|Q(p^{-n/2}x) - f(p^{-n/2}x)\| + \|Q'(p^{-n/2}x) - f(p^{-n/2}x)\|\right\} \\ &\leq p^n \frac{2c}{p^{r/2} - p}\|p^{-n/2}x\|^r = p^{n(2-r)/2}\frac{2c}{p^{r/2} - p}\|x\|^r \to 0, \\ &\quad \text{as } n \to \infty, \end{aligned} \tag{2.25}$$

if I_4 holds: $p^{2-r} < 1$.

Thus from (2.22), (2.23), (2.24) and (2.25) we find $Q(x) = Q'(x)$ for all $x \in X$.

This completes the proof of the *uniqueness* and *stability* of the quadratic functional equation (*). □

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PEDAGOGICAL DEPARTMENT, E. E.
NATIONAL AND CAPODISTRIAN UNIVERSITY OF ATHENS
SECTION OF MATHEMATICS AND INFORMATICS
4, AGAMEMNONOS STR.
AGHIA PARASKEVI, ATHENS 15342
GREECE
jrassias@primedu.uoa.gr; jrassias@tellas.gr

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BASICITY OF WEIGHTED SHIFT OPERATORS ON LOCALLY CONVEX SPACES

M. MALDONADO AND J. PRADA

Abstract. Conditions for a weighted shift to be a basis operator on Köthe spaces are given.

1. Introduction

Weighted shift operators on Hilbert space, on ℓ^p and Banach spaces have been studied in many papers such as [2], [4], [5], [6], [8], [11], [13], [14], [15], [17], [19], [20]. The relationship between weighted shifts and Banach algebras has mainly been treated by Grabiner [7].

In [14] the concept of a basis operator on the space ℓ^p, $1 \leq p < \infty$, or c_0 is introduced and criteria for a weighted shift operator to be a basis operator are given. In this paper similar techniques to those used in [14] allow us to deduce necessary and sufficient conditions for a weighted shift operator to be a basis operator in the more general context of certain locally convex spaces, precisely Köthe spaces (such as infinite and finite power series spaces that include, in particular, the space of analytic functions on a disc) which are projective limits of Banach sequence spaces.

2. Terminology and basic results

Denote by $\lambda^p(A)$, $1 \leq p < \infty$, the Köthe (echelon) space given by the matrix $A = ((a_{n,k}))$, $n, k = 0, 1, 2, \dots$, $a_{n,k} > 0$, $a_{n,k} \leq a_{n,k+1}$, for all k, n, that is,

$$\lambda^p(A) = \left\{ x = (x_n)_{n=0}^{\infty},\, x_n \in \mathbb{C}, \sum_{n=0}^{\infty} (|x_n|\, a_{n,k})^p < \infty,\, \forall\, k = 0, 1, 2, \dots \right\}$$

with the norms

$$\|x\|_k = \left(\sum_{n=0}^{\infty} (|x_n|\, a_{n,k})^p \right)^{\frac{1}{p}}$$

so that $\lambda^p(A)$ is a Fréchet space. If $p = \infty$, $\lambda^\infty(A)$ is

$$\lambda^\infty(A) = \left\{ x = (x_n)_{n=0}^{\infty},\, x_n \in \mathbb{C}, \sup_n (|x_n|\, a_{n,k}) < \infty,\, \forall\, k = 0, 1, 2, \dots \right\}$$

and

$$\lambda^0(A) = \left\{ x = (x_n)_{n=0}^{\infty},\, x_n \in \mathbb{C}, \lim_n |x_n|\, a_{n,k} = 0,\, \forall\, k = 0, 1, 2, \dots \right\}.$$

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We define $\ell^p(\{a_n\})$ (respectively $c_0(\{a_n\})$), the weighted ℓ^p (respectively c_0) space corresponding to a positive sequence $\{a_n\}$ as

$$\ell^p(\{a_n\}) = \{x = (x_n)_{n=0}^{\infty} : (x_n a_n)_{n=0}^{\infty} \in \ell^p\}\,.$$
$$c_0(\{a_n\}) = \{x = (x_n)_{n=0}^{\infty} : (x_n a_n)_{n=0}^{\infty} \in c_0\}\,.$$

Observe that in case $a_{n,k} = a_n$ for all k, then $\ell^p(\{a_n\}) = \lambda^p(A)$.

$\lambda^p(A)$ (respectively $\lambda^0(A)$) is the projective limit of the weighted ℓ^p spaces $\ell^p(\{a_{n,k}\})$ (respectively $c_0(\{a_{n,k}\})$) corresponding to the sequence $\{a_{n,k}\}_n$, with the usual topology.

The canonical basis in these spaces is denoted by $\delta_n = (\delta_{n,k})_{k=0}^{\infty}$, where $\delta_{n,k}$ is the Kronecker delta; $\varphi = \operatorname{span}\{\delta_n : n \in \mathbb{N}_0\}$.

The space $\lambda^1(A)$ is nuclear if and only if

$$\forall\, k,\ \exists\, N(k) \text{ such that } \left(\frac{a_{n,k}}{a_{n,N(k)}}\right) \in \ell^1$$

[12] and then $\lambda^p(A) = \lambda^1(A) = \lambda^0(A)$, $p \in [1, \infty)$.

Well-known examples of echelon spaces are the infinite and finite power series spaces; in the first type, the matrix is $(a_{n,k} = e^{k\alpha_n})$ while in the second $(a_{n,k} = e^{-\frac{1}{k}\alpha_n})$, where (α_n) is a monotonically increasing sequence of real numbers going to infinity. Examples of nuclear infinite (respectively finite) power series spaces are $\mathcal{H}(\mathbb{C})$, the space of entire functions on the complex plane and $\mathcal{H}(\mathbb{D})$, the space of holomorphic functions on the unit disc (in both cases $\alpha_n = n$). When $\alpha_n = \log n$, the corresponding nuclear infinite power series space is very well-known, in fact, the space s of sequences rapidly decreasing to zero.

A linear map T from $\lambda^p(A)$ to $\lambda^p(A)$ is continuous if and only if for all $k \in \mathbb{N}$ there exist $r = r(k) \in \mathbb{N}$ and $C = C_k > 0$ such that

$$\|Tx\|_k \le C\,\|x\|_r\,, \quad \forall\ x \in \lambda^p(A).$$

If T is a continuous operator from $\lambda^p(A)$ to $\lambda^p(A)$, then for all $k \in \mathbb{N}$, there exists $r \in \mathbb{N}$ such that T extends into a continuous operator from $\ell^p(\{a_{n,r}\})$ to $\ell^p(\{a_{n,k}\})$.

Denote by S the shift operator

$$S(x) = S(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots)\,.$$

formally

$$S\Big(\sum_{n=0}^{\infty} x_n z^n\Big) = \sum_{n=0}^{\infty} x_n z^{n+1}\,;$$

and by T the weighted shift operator

$$T(x_0, x_1, x_2 \dots) = (0, \lambda_0 x_0, \lambda_1 x_1, \lambda_2 x_2, \dots)\,;$$

that is,

$$T\Big(\sum_{n=0}^{\infty} x_n z^n\Big) = \sum_{n=0}^{\infty} \lambda_n x_n z^{n+1}\,,$$

where the sequence (λ_n) is taken of real positive numbers to simplify computations. It is assumed that S and T are continuous operators on $\lambda^p(A)$.

By $\lambda^p_m(A)$, $m = 0, 1, 2, \dots$, $p \in [1, \infty]$ are denoted the following subspaces of $\lambda^p(A)$ (which are Fréchet spaces also):

$$\lambda^p_m(A) = \{x = (x_n)_{n=0}^{\infty} \in \lambda^p(A) \text{ such that } x_n = 0,\, n < m\}\,.$$

Analogously, we define

$$\ell^p_m(\{a_n\}) = \{x = (x_n)_{n=0}^{\infty} \in \ell^p(\{a_n\}) \text{ such that } x_n = 0,\, n < m\}\,.$$

$$c_{0,m}(\{a_n\}) = \{x = (x_n)_{n=0}^{\infty} \in c_0(\{a_n\}) \text{ such that } x_n = 0,\, n < m\}\,.$$

The concept of basis operator is central in this work. In [14] the following definition is given,

Definition (2.1). Let T be a linear operator on the space ℓ^p. T is a basis operator if and only if it is cyclic and for any $x \neq 0$, $x \in \ell^p$, there exists a linear isomorphism V of ℓ^p onto itself, an integer i, $i \geq 0$, and a sequence of complex numbers $(t_n)_{n=0}^{\infty}$ such that $V\delta_{n+i} = t_n T^n(x)$, $n \geq 0$.

A characterization of basis operators was given by Nikol'skii [14] in the following Lemma,

LEMMA (2.2). *A weighted shift operator T with $r(T) = \liminf |\lambda_0 \dots \lambda_n|^{1/n} = 0$ is a basis operator in ℓ^p, $1 \leq p < \infty$, if and only if for any $x = (x_n)$, $x \neq 0$, $x \in \ell^p$, the operator $V(x)$ defined on the unit vectors δ_n, $n \geq 1$, by the equations*

$$V(x)\delta_n = \delta_n, \qquad n < i(x)\,,$$

$$V(x)\delta_i = \delta_i + \sum_{j=i+1}^{\infty} \frac{x_j}{x_i}\,\delta_j\,,$$

$$V(x)\delta_n = \delta_n + \sum_{j=i+1}^{\infty} \frac{\lambda_j \dots \lambda_{j+n-i-1}}{\lambda_i \dots \lambda_{n-1}}\,\frac{x_j}{x_i}\,\delta_{j+n-i}, \qquad n > i(x),$$

with $i = i(x) = \min\{n : x_n \neq 0\}$, extends (by linearity) to an isomorphism of the space ℓ^p_i onto itself.

Remark (2.3). Note that Definition (2.1) makes sense on c_0 and that Lemma (2.2) remains in effect for c_0 as well.

We are going to take the previous result, adapted to locally convex spaces, as a definition, which is enough for our purposes.

The concept of basicity on ℓ^p spaces is strongly related to various problems in function theory, such as the study of invariant subspaces of weighted shift operators, unicellularity of linear operators and the existence of Riesz bases for the subspaces ℓ^p_n of ℓ^p.

The condition $r(T) = \liminf |\lambda_0 \dots \lambda_n|^{1/n} = 0$ plays an important role in the investigation of basis operators. When $r(T) = 0$ the basis operator T is unicellular and the next corollary follows [14].

COROLLARY (2.4). *If T is a basis weighted shift operator in the space ℓ^p, $1 \leq p < \infty$, or in c_0, if $r(T) = 0$ and the triple $\{V, i, \{t_n\}_{n=0}^{\infty}\}$ corresponds to an element $x = (x_n) \neq 0$, according to the definition of basicity, then $i = i(x)$ is the smallest number n for which $x_n \neq 0$.*

If $r(T) > 0$ it is an open question (as far as we know) if the previous result is true.

Definition (2.5). Let T be a weighted shift operator on $\lambda^p(A)$, $p \in \{0\} \cup [1, \infty)$; T is a basis operator if and only if for any $x \neq 0$, $x \in \lambda^p(A)$, the operator $V(x)$ defined by

$$V(x)\delta_n = \delta_n, \qquad n < i(x),$$

$$V(x)\delta_i = \delta_i + \sum_{j=i+1}^{\infty} \frac{x_j}{x_i}\,\delta_j,$$

$$V(x)\delta_n = \delta_n + \sum_{j=i+1}^{\infty} \frac{\lambda_j \cdots \lambda_{j+n-i-1}}{\lambda_i \cdots \lambda_{n-1}} \frac{x_j}{x_i}\,\delta_{j+n-i}, \qquad n > i(x)$$

gives an isomorphism from $\lambda_i^p(A)$ to $\lambda_i^p(A)$ $(i = i(x) = \min\{n : x_n \neq 0\})$.

This definition can be given in the same way for two different Köthe spaces.

Remark (2.6). If we consider two weighted ℓ^p spaces, $p \in [1, \infty)$ (when $p = \infty$ take c_0), following the commutative diagram

$$\begin{array}{ccc} \ell^p(\{a_{n,r}\}) & \xrightarrow{V(x)} & \ell^p(\{a_{n,k}\}) \\ {\scriptstyle F_r}\downarrow & & \downarrow{\scriptstyle F_k} \\ \ell^p & \xrightarrow{V} & \ell^p \end{array}$$

with $F_j(x_n) = (x_n a_{n,j})$, $j = 0, 1, \ldots$ we can observe that the weighted shift operator T from $\ell^p(\{a_{n,r}\})$ to $\ell^p(\{a_{n,k}\})$ is a basis operator if and only if the operator

$$V\delta_n = \frac{a_{n,k}}{a_{n,r}}\,\delta_n, \qquad n < i(y),$$

$$V\delta_i = \frac{a_{i,k}}{a_{i,r}}\,\delta_i + \sum_{j=i+1}^{\infty} \frac{y_j}{y_i} \frac{a_{j,k}}{a_{j,r}}\,\delta_j,$$

$$V\delta_n = \frac{a_{n,k}}{a_{n,r}}\,\delta_n + \frac{a_{i,r}}{a_{n,r}} \sum_{j=i+1}^{\infty} \frac{\lambda_j \cdots \lambda_{j+n-i-1}}{\lambda_i \cdots \lambda_{n-1}} \frac{y_j}{y_i} \frac{a_{j+n-i,k}}{a_{j,r}}\,\delta_{j+n-i}, \qquad n > i(y)$$

where $(y_n) = (x_n a_{n,r})$, is an isomorphism of the space ℓ_i^p onto itself.

Observe also that if $r = k$, then T is a basis operator from $\ell^p(\{a_{n,k}\})$ to $\ell^p(\{a_{n,k}\})$ if and only if $\Phi(x) = (0, \lambda_n \frac{a_{n+1,k}}{a_{n,k}} x_n)$ is a basis operator from ℓ^p to ℓ^p, while if $k \neq r$ both conditions are not, in general, equivalent; in fact, the operator $\Phi(x) = (0, \lambda_n \frac{a_{n+1,k}}{a_{n,r}} x_n)$ from ℓ^p to ℓ^p is a basis operator if and only if, for all $x \in \ell^p$, the operator

$$V\delta_n = \delta_n + \sum_{j=i+1}^{\infty} \frac{\lambda_j \cdots \lambda_{j+n-i-1}}{\lambda_i \cdots \lambda_{n-1}} \frac{a_{j+1,k} \cdots a_{j+n-i,k}}{a_{i+1,k} \cdots a_{n,k}} \frac{a_{i,r} \cdots a_{n-1,r}}{a_{j,r} \cdots a_{j+n-i-1,r}} \frac{x_j}{x_i}\,\delta_{j+n-i}, \qquad n > i,$$

$$V\delta_i = \delta_i + \sum_{j=i+1}^{\infty} x_j \delta_j$$

is an isomorphism from ℓ_i^p onto ℓ_i^p.

The next Lemma about linear operators between ℓ^p spaces is easily extended to weighted ℓ^p spaces; it is stated below and will be used later on.

LEMMA (2.7). *If B is a linear operator from ℓ^1 to ℓ^1 given by the matrix $(t_{m,q})_{m,q=0}^{\infty}$, then*

$$\|B\| = \sup_{q\geq 0}\left\{\sum_{m=0}^{\infty}|t_{m,q}|\right\}.$$

If B is a linear operator from ℓ^∞ to ℓ^∞ (respectively from c_0 to c_0) with matrix $(t_{m,q})_{m,q=0}^{\infty}$, then

$$\|B\| = \sup_{m\geq 0}\left\{\sum_{q=0}^{\infty}|t_{m,q}|\right\}.$$

If B is a linear operator from ℓ^1 to c_0 given by the matrix $(t_{i,j})_{i,j=0}^{\infty}$, then

$$\|B\| = \sup_{i,j\geq 0}\{|t_{i,j}|\}.$$

LEMMA (2.8). *If B is a continuous linear operator from $\ell^1(\{a_{n,r}\})$ to $\ell^1(\{a_{n,k}\})$ given by the matrix $(t_{m,q})_{m,q=0}^{\infty}$, then*

$$\|B\| = \sup_{q\geq 0}\left\{\sum_{m=0}^{\infty}|t_{m,q}|\frac{a_{m,k}}{a_{q,r}}\right\}.$$

If B is a continuous linear operator from $c_0(\{a_{n,r}\})$ to $c_0(\{a_{n,k}\})$ given by the matrix $(t_{m,q})_{m,q=0}^{\infty}$, then

$$\|B\| = \sup_{m\geq 0}\left\{\sum_{q=0}^{\infty}|t_{m,q}|\frac{a_{m,k}}{a_{q,r}}\right\}.$$

If B is a continuous linear operator from $\ell^1(\{a_{n,r}\})$ to $c_0(\{a_{n,k}\})$ given by the matrix $(t_{i,j})_{i,j=0}^{\infty}$, then

$$\|B\| = \sup_{i,j\geq 0}\left\{|t_{i,j}|\frac{a_{i,k}}{a_{j,r}}\right\}.$$

3. Basis operators on Köthe spaces

Some extra notation will be needed: if $x = (x_n) \in \lambda^p(A)$, the coordinate projections are denoted by P_n; that is,

$$P_n x = x_n, \quad n \geq 0.$$

For $x \in \lambda_i^p(A)$, that is, $P_n x = 0$, $n < i$ and $P_i x \neq 0$ $(i \geq 0)$ we set

$$\Gamma_i(x) = (V(x) - I)P_i x.$$

This operator $\Gamma_i(x)$ has the following properties:

1. It is a linear operator for any i, given by the formulas

$$\begin{aligned}
\Gamma_i(x)\delta_n &= \sum_{j=i+1}^{\infty} \frac{\lambda_j \cdots \lambda_{j+n-i-1}}{\lambda_i \cdots \lambda_{n-1}}\, x_j \delta_{j+n-i}\,, && n > i\,,\\
\Gamma_i(x)\delta_i &= \sum_{j=i+1}^{\infty} x_j \delta_j\,, \\
\Gamma_i(x)\delta_n &= 0\,, && n < i\,.
\end{aligned}$$

2. $\Gamma_i(\lambda x + \mu y) = \lambda \Gamma_i(x) + \mu \Gamma(y)$, x, $y \in \lambda^p(A)$, λ, $\mu \in \mathbb{C}$. Then, for a fixed $i \geq 0$, the mapping Γ defined by $\Gamma(x) = \Gamma_i(x)$, for all $x \in \lambda_i^p(A)$ is again a linear mapping from $\lambda^p(A)$ to $L(\varphi, \varphi)$.

3. For all $x \in \lambda_i^p(A)$ the matrix $(t_{m,q}(x))_{m,q=0}^{\infty}$ of the operator $\Gamma_i(x)$ is

$$t_{m,q}(x) = \begin{cases} 0 & \text{if } m \leq q\,,\\ 0 & \text{if } 0 \leq q \leq i-1\,,\\ x_m & \text{if } q = i \text{ and } m \geq i+1\,,\\ \frac{\lambda_{m-q+i} \cdots \lambda_{m-1}}{\lambda_i \cdots \lambda_{q-1}}\, x_{m-q+i} & \text{if } q \geq i+1 \text{ and } m \geq q+1. \end{cases}$$

As the Köthe space $\lambda^p(A)$, $p = 0$, $1 \leq p < \infty$, is the projective limit of the Banach spaces $\ell^p(\{a_{n,k}\})$, we have

Theorem (3.1). *If T is a basis operator on $\lambda^1(A)$, then $\forall\, k \in \mathbb{N}$, $\exists\, r = r(k) \in \mathbb{N}$, $r \geq k$ and $s = s(k) \in \mathbb{N}$ such that*

$$K_i = \sup_{j \geq 1} \sup_{q \geq i+1} \left\{ \frac{\lambda_{j+i} \cdots \lambda_{j+q-1}}{\lambda_i \cdots \lambda_{q-1}} \frac{a_{q+j,k}}{a_{q,r}\, a_{i+j,s}} \right\} < \infty$$

for all $i = 0, 1, 2, \ldots$

Proof. Fix an integer i, $i \geq 0$; let us consider the linear map Γ from $\lambda_{i+1}^1(A)$ to $\mathcal{L}(\lambda^1(A))$ (the space of all linear continuous operators on $\lambda^1(A)$) taking an element $x \in \lambda_{i+1}^1(A)$ into the operator $\Gamma_i(\delta_i + x)$.

If $x \in \lambda_{i+1}^1(A)$, as $\Gamma_i(\delta_i + x) \in \mathcal{L}(\lambda^1(A))$, then for every $k \in \mathbb{N}$ there exist $r_x(k) \in \mathbb{N}$ and $C_{x,k} > 0$ so that

$$\|\Gamma_i(\delta_i + x)\delta_m\|_k \leq C_{x,k}\, \|\delta_m\|_{r_x(k)}$$

for all $m = 0, 1, 2, \ldots$

Let $k \in \mathbb{N}$ be a fixed number and for every $r \in \mathbb{N}$, $r \geq k$ consider the linear subspaces

$$X_{i,r} = \left\{ x \in \lambda_{i+1}^1(A) \text{ such that } r_x(k) = r \right\}.$$

Therefore if $x \in X_{i,r}$, the mapping $\Gamma_i(\delta_i + x)$ from $\ell^1(\{a_{n,r}\})$ to $\ell^1(\{a_{n,k}\})$ is continuous.

As $\lambda_{i+1}^1(A)$ is a Baire space (Proposition 1.2.9 of [16]) and $\lambda_{i+1}^1(A) = \bigcup_{r \in \mathbb{N}, r \geq k} X_{i,r}$, there exists $r \in \mathbb{N}$, $r \geq k$ such that $X_{i,r}$ is a Baire space dense in $\lambda_{i+1}^1(A)$ (Proposition 1.2.3. of [16]).

From the Closed Graph Theorem (Theorem 1.2.19 of [16]), the linear map

$$\Gamma\colon \lambda_{i+1}^1(A) \longrightarrow \mathcal{L}\left(\ell^1(\{a_{n,r}\}), \ell^1(\{a_{n,k}\})\right)$$

is continuous: if $\{x^n\}_{n\in\mathbb{N}}$ is a sequence that converges to x in $\lambda^1_{i+1}(A)$ and $\Gamma_i(\delta_i + x^n)$ converges to an operator Υ in the operator norm, then

$$(\Upsilon\delta_q)_m = \left(\lim_{n\to\infty} \Gamma_i(\delta_i + x^n)\delta_q\right)_m = (\Gamma_i(\delta_i + x)\delta_q)_m$$

for $q \geq i$ and $m \geq i+1$, that is $\Upsilon = \Gamma_i(\delta_i + x)$.

So $X_{i,r} = \lambda^1_{i+1}(A)$ and moreover, there exist $s = s(k) \in \mathbb{N}$ and $C = C(k) > 0$ so that

$$\|\Gamma_i(\delta_i + x)\| \leq C\,\|x\|_s\,, \quad \text{for all } x \in \lambda^1_{i+1}(A).$$

By Lemma (2.8),

$$\|\Gamma_i(\delta_i + x)\| = \sup_{q\geq i+1}\left\{\sum_{m=i+1}^{\infty} |x_m|\,\frac{a_{m,k}}{a_{i,r}},\ \sum_{m=q+1}^{\infty} \frac{\lambda_{m-q+i}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{q-1}}\,|x_{m-q+i}|\,\frac{a_{m,k}}{a_{q,r}}\right\}$$

$$\geq \sup_{q\geq i+1}\left\{\sum_{j=1}^{\infty} \frac{\lambda_{j+i}\cdots\lambda_{j+q-1}}{\lambda_i\cdots\lambda_{q-1}}\,|x_{j+i}|\,\frac{a_{j+q,k}}{a_{q,r}}\right\}.$$

Then

$$\sup_{q\geq i+1}\left\{\sum_{j=1}^{\infty} \frac{\lambda_{j+i}\cdots\lambda_{j+q-1}}{\lambda_i\cdots\lambda_{q-1}}\,|x_{j+i}|\,\frac{a_{j+q,k}}{a_{q,r}}\right\} \leq \|\Gamma_i(\delta_i + x)\| \leq C\,\|x\|_s\,.$$

Taking $x^j = \frac{1}{a_{i+j,s}}\delta_{i+j}$, $j = 1, 2, 3, \ldots$ in the previous inequality, it follows that

$$\sup_{j\geq 1}\sup_{q\geq i+1}\left\{\frac{\lambda_{j+i}\cdots\lambda_{j+q-1}}{\lambda_i\cdots\lambda_{q-1}}\,\frac{a_{j+q,k}}{a_{q,r}\,a_{i+j,s}}\right\} < \infty,$$

for all $i = 0, 1, 2, \ldots$, or equivalently

$$\sup_{q\geq i+1}\sup_{m\geq q+1}\left\{\frac{\lambda_{m-q+i}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{q-1}}\,\frac{a_{m,k}}{a_{q,r}\,a_{m-q+i,s}}\right\} < \infty,$$

for all $i = 0, 1, 2, \ldots$ □

Theorem (3.2). *If*

$$\lim_{n\to\infty} \lambda_n \frac{a_{n+1,k}}{a_{n,k}} = 0$$

and $\forall\, k \in \mathbb{N}$, $\exists\, r = r(k) \in \mathbb{N}$ *such that*

$$K_i = \sup_{q\geq i+1}\sup_{m\geq q+1}\left\{\frac{\lambda_{m-q+i}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{q-1}}\,\frac{a_{m,k}}{a_{q,k}a_{m-q+i,r}}\right\} < \infty$$

for all $i = 0, 1, 2, \ldots$, *then the operator* T *is a basis operator on* $\lambda^1(A)$.

Proof. Let $i \geq 0$ be a fixed integer. For every $k \in \mathbb{N}$ we have

$$\sup_{m\geq q+1}\left\{\frac{\lambda_{m-q+i}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{q-1}}\,\frac{a_{m,k}}{a_{q,k}\,a_{m-q+i,r}}\right\}$$

$$\leq \frac{\lambda_q}{\lambda_i}\frac{a_{q+1,k}}{a_{q,k}}\sup_{m\geq q+2}\left\{\frac{1}{a_{i+1,r}},\ \frac{\lambda_{m-(q+1)+(i+1)}\cdots\lambda_{m-1}}{\lambda_{i+1}\cdots\lambda_q}\,\frac{a_{m,k}}{a_{q+1,k}\,a_{m-(q+1)+(i+1),r}}\right\}.$$

As $\lim_{n\to\infty} \lambda_n \frac{a_{n+1,k}}{a_{n,k}} = 0$, then

$$\sup_{q\geq n}\sup_{m\geq q+1}\left\{\frac{\lambda_{m-q+i}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{q-1}}\frac{a_{m,k}}{a_{q,k}\,a_{m-q+i,r}}\right\}$$

$$\leq \sup_{q\geq n}\left\{\frac{\lambda_q}{\lambda_i}\frac{a_{q+1,k}}{a_{q,k}}\right\}\left(\frac{1}{a_{i+1,r}}+K_{i+1}\right)\overset{n\to\infty}{\longrightarrow} 0.$$

Therefore if $x \in \lambda_i^1(A)$ with $x_i = 1$, there exists $n \in \mathbb{N}$ such that

$$\sup_{q\geq n}\sup_{m\geq q+1}\left\{\frac{\lambda_{m-q+i}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{q-1}}\frac{a_{m,k}}{a_{q,k}\,a_{m-q+i,r}}\right\}\|x\|_r < 1.$$

Consider now the operator $\Gamma_i(x)$, which is continuous from $\ell_n^1(\{a_{m,k}\})$ to itself for any $k \in \mathbb{N}$, since for $q \geq n$,

$$\|\Gamma_i(x)\delta_q\|_k \leq \sup_{m\geq q+1}\left\{\frac{\lambda_{m-q+i}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{q-1}}\frac{a_{m,k}}{a_{q,k}\,a_{m-q+i,r}}\right\}\|\delta_q\|_k\sum_{m=q+1}^{\infty}|x_{m-q+i}|\,a_{m-q+i,r}$$

$$\leq K_i\,\|\delta_q\|_k\,\|x\|_r\,.$$

Then the norm of the restriction of $\Gamma_i(x)$ to $\ell_n^1(\{a_{m,k}\})$ satisfies

$$\|\Gamma_i(x)\| \leq \sup_{q\geq n}\sup_{m\geq q+1}\left\{\frac{\lambda_{m-q+i}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{q-1}}\frac{a_{m,k}}{a_{q,k}\,a_{m-q+i,r}}\right\}\|x\|_r < 1.$$

The restriction of the operator $V(x) = I + \Gamma_i(x)$ to $\ell_n^1(\{a_{m,k}\})$ is an isomorphism, so $V(x)$ is an isomorphism from $\ell_i^1(\{a_{m,k}\})$ onto itself, for every k, and the Theorem is proved. □

Similarly to the two previous Theorems we have

Theorem (3.3). *If T is a basis operator on $\lambda^0(A)$, then the following condition is true:*

$\forall\, k \in \mathbb{N}$, $\exists\, r = r(k) \in \mathbb{N}$, $r \geq k$ and $s = s(k) \in \mathbb{N}$ such that

$$\sup_{m\geq i+2}\left\{\sum_{j=0}^{m-i-2}\frac{\lambda_{m-j-1}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{i+j}}\frac{a_{m,k}}{a_{j+i+1,r}a_{m-j-1,s}}\right\}<\infty$$

for all $i = 0, 1, 2, \ldots$

Theorem (3.4). *If*

$$\lim_{n\to\infty}\lambda_n\frac{a_{n+1,k}}{a_{n,k}} = 0$$

and $\forall\, k \in \mathbb{N}$, $\exists\, r = r(k) \in \mathbb{N}$, $r \geq k$, such that

$$K_i = \sup_{m\geq i+2}\left\{\sum_{j=0}^{m-i-2}\frac{\lambda_{m-j-1}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{i+j}}\frac{a_{m,k}}{a_{j+i+1,k}a_{m-j-1,r}}\right\}<\infty$$

for all $i = 0, 1, 2, \ldots$, then the operator T is a basis operator on $\lambda^0(A)$.

We can extend these results to $\lambda^p(A)$ as follows.

Theorem (3.5). *If T is a basis operator on $\lambda^p(A)$, $p=0$, $1\le p<\infty$, then $\forall\ k\in\mathbb{N}$, $\exists\, r=r(k)\in\mathbb{N}$, $r\ge k$ and $s=s(k)\in\mathbb{N}$ such that*

$$\sup_{j\ge 1}\sup_{q\ge i+1}\left\{\frac{\lambda_{j+i}\cdots\lambda_{j+q-1}}{\lambda_i\cdots\lambda_{q-1}}\frac{a_{q+j,k}}{a_{q,r}\,a_{i+j,s}}\right\}<\infty$$

for all $i=0,1,2,\ldots$

Proof. For a fixed integer i, $i\ge 0$, we have the linear map

$$\begin{aligned}\Gamma\colon \lambda^p_{i+1}(A) &\longrightarrow \mathfrak{L}\left(\lambda^p(A),\lambda^p(A)\right)\\ x &\longmapsto \Gamma(x)=\Gamma_i(\delta_i+x).\end{aligned}$$

As $\Gamma_i(\delta_i+x)$, $x\in\lambda^p_{i+1}(A)$, is a linear and continuous operator, then $\forall\ k$ there exist $r_x(k)$ and $C_{x,k}>0$ such that

$$\left\|\Gamma_i(\delta_i+x)\delta_m\right\|_k\le C_{x,k}\left\|\delta_m\right\|_{r_x(k)},\qquad \forall\, m=0,1,2,\ldots$$

For a fixed $k\in\mathbb{N}$, for every $r\in\mathbb{N}$, $r\ge k$, let us consider

$$X_{i,r}=\left\{x\in\lambda^p_{i+1}(A)\text{ such that }r_x(k)=r\right\}.$$

Therefore if $x\in X_{i,r}$, the mapping

$$\Gamma_i(\delta_i+x)\colon \ell^p(\{a_{n,r}\})\longrightarrow \ell^p(\{a_{n,k}\})$$

is linear and continuous, so the restriction of $\Gamma_i(\delta_i+x)$ to $\ell^1(\{a_{n,r}\})$ is continuous too.

As $\lambda^p_{i+1}(A)$ is a Baire space and $\lambda^p_{i+1}(A)=\bigcup_{r\in\mathbb{N},r\ge k}X_{i,r}$, by Proposition 1.2.3. of [16], there exists $r\in\mathbb{N}$ such that $X_{i,r}$ is a Baire space dense in $\lambda^p_{i+1}(A)$.

As in Theorem (3.1), from the Closed Graph Theorem, the linear map

$$\Gamma\colon \lambda^p_{i+1}(A)\longrightarrow \mathfrak{L}\left(\ell^p(\{a_{n,r}\}),\ell^p(\{a_{n,k}\})\right)$$

is continuous. Then there exist $s=s(k)\in\mathbb{N}$ and $C>0$ such that

$$\left\|\Gamma_i(\delta_i+x)\right\|_{p,p}\le C\left\|x\right\|_{s,p},$$

and it follows that

$$\left\|\Gamma_i(\delta_i+x)\right\|_{1,\infty}\le C\left\|x\right\|_{s,p}.$$

By Lemma (2.8),

$$\begin{aligned}&\left\|\Gamma_i(\delta_i+x)\right\|_{1,\infty}\\ &=\sup_{q\ge i+1}\left\{\sup_{m\ge i+1}\left\{|x_m|\,\frac{a_{m,k}}{a_{i,r}}\right\},\ \sup_{m\ge q+1}\left\{\frac{\lambda_{m-q+i}\ldots\lambda_{m-1}}{\lambda_i\ldots\lambda_{q-1}}\,|x_{m-q+i}|\,\frac{a_{m,k}}{a_{q,r}}\right\}\right\}\\ &\ge\sup_{\substack{q\ge i+1\\ m\ge q+1}}\left\{\frac{\lambda_{m-q+i}\ldots\lambda_{m-1}}{\lambda_i\ldots\lambda_{q-1}}\,|x_{m-q+i}|\,\frac{a_{m,k}}{a_{q,r}}\right\}\\ &=\sup_{\substack{q\ge i+1\\ j\ge 1}}\left\{\frac{\lambda_{j+q-1}\ldots\lambda_{j+i}}{\lambda_i\ldots\lambda_{q-1}}\,|x_{j+i}|\,\frac{a_{j+q,k}}{a_{q,r}}\right\}.\end{aligned}$$

Then

$$\sup_{\substack{q\geq i+1\\ j\geq 1}}\left\{\frac{\lambda_{j+q-1}\dots\lambda_{j+i}}{\lambda_i\dots\lambda_{q-1}}\,|x_{j+i}|\,\frac{a_{j+q,k}}{a_{q,r}}\right\}\leq\|\Gamma_i(\delta_i+x)\|_{1,\infty}\leq C\,\|x\|_{s,p}\,.$$

Taking $x^j=\frac{1}{a^s_{j+i}}\,\delta_{j+i},\ j=1,2,3,\dots,$ in the previous inequality, we obtain the result. □

Theorem (3.6). *If*

$$\lim_{n\to\infty}\lambda_n\frac{a_{n+1,k}}{a_{n,k}}=0$$

and $\forall\,k\in\mathbb{N}$

$$K_i=\sup_{m\geq i+2}\left\{\sum_{j=0}^{m-i-2}\frac{\lambda_{m-j-1}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{i+j}}\,\frac{a_{m,k}}{a_{j+i+1,k}a_{m-j-1,k}}\right\}<\infty \tag{3.7}$$

for all $i=0,1,2,\dots,$ *then the weighted shift* T *is a basis operator on* $\lambda^p(A)$.

Proof. Condition (3.7) clearly implies

$$C_i=\sup_{q\geq i+1}\sup_{m\geq q+1}\left\{\frac{\lambda_{m-q+i}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{q-1}}\,\frac{a_{m,k}}{a_{q,k}\,a_{m-q+i,k}}\right\}<\infty.$$

We consider the application $B\colon\lambda^1(A)\times\lambda^1(A)\longrightarrow\lambda^1(A)$ defined by $B(x,y)=\Gamma_i(x)(y),\ i=i(x)=\min\{n:x_n\neq 0\}$.

If $x=(x_n),\ y=(y_n)\in\lambda^p(A)$ and $k\in\mathbb{N}$, we have

$$\begin{aligned}\|B(x,y)\|_{k,1}&=\|\Gamma_i(x)y\|_{k,1}\\&=\left\|y_i\sum_{j=i+1}^{\infty}x_j\delta_j+\sum_{q=i+1}^{\infty}y_q\sum_{m=q+1}^{\infty}\frac{\lambda_{m-q+i}\dots\lambda_{m-1}}{\lambda_i\dots\lambda_{q-1}}\,x_{m-q+i}\delta_m\right\|_{k,1}\\&\leq|y_i|\sum_{j=i+1}^{\infty}|x_j|\,a_{j,k}+\sum_{q=i+1}^{\infty}|y_q|\sum_{m=q+1}^{\infty}\frac{\lambda_{m-q+i}\dots\lambda_{m-1}}{\lambda_i\dots\lambda_{q-1}}\,|x_{m-q+i}|\,a_{m,k}\\&\leq\left(\frac{1}{a_{i,k}}+C_i\right)\|x\|_{k,1}\,\|y\|_{k,1}\,.\end{aligned}$$

Therefore, for any $k\in\mathbb{N}$, the above bilinear mapping B extends to a bilinear map

$$B\colon\ell^1(\{a_{m,k}\})\times\ell^1(\{a_{m,k}\})\longrightarrow\ell^1(\{a_{m,k}\}),$$

which verifies that for all $x\in\ell^1_i(\{a_{m,k}\})$ and $y\in\ell^1(\{a_{m,k}\})$

$$\|B(x,y)\|_{k,1}=\|\Gamma_i(x)y\|_{k,1}\leq M_i\,\|x\|_{k,1}\,\|y\|_{k,1}\,,\qquad M_i=\frac{1}{a_{i,k}}+C_i.$$

Proceeding as in Theorem (3.2) we obtain that for $k\in\mathbb{N}$, $x\in\ell^1_i(\{a_{m,k}\})$ and $y\in\ell^1_n(\{a_{m,k}\}),\ n>i$

$$\|B(x,y)\|_{k,1}=\|\Gamma_i(x)y\|_{k,1}\leq C_n\,\|x\|_{k,1}\,\|y\|_{k,1}\overset{n\to\infty}{\longrightarrow}0.$$

Analogously, if we consider $B\colon \lambda^0(A) \times \lambda^0(A) \longrightarrow \lambda^0(A)$ defined as before, for any $k \in \mathbb{N}$ it extends to a bilinear mapping

$$B\colon c_0(\{a_{m,k}\}) \times c_0(\{a_{m,k}\}) \longrightarrow c_0(\{a_{m,k}\}),$$

such that if $x \in c_{0,i}(\{a_{m,k}\})$ and $y \in c_0(\{a_{m,k}\})$, then

$$\|B(x,y)\|_{k,\infty} \leq M_i' \|x\|_{k,\infty} \|y\|_{k,\infty}\,, \qquad M_i' = \frac{1}{a_{i,k}} + K_i.$$

Furthermore, if $x \in c_{0,i}(\{a_{m,k}\})$ and $y \in c_{0,n}(\{a_{m,k}\})$,

$$\|B(x,y)\|_{k,\infty} \leq K_n \|x\|_{k,\infty} \|y\|_{k,\infty}\,.$$

By the Interpolation Theorem (Theorem 2.7 of [1]), B can be extended to a bilinear operator

$$B\colon \ell^p(\{a_{m,k}\}) \times \ell^p(\{a_{m,k}\}) \longrightarrow \ell^p(\{a_{m,k}\}),$$

with

$$\|B(x,y)\|_{k,p} \leq M_i^t (M_i')^{1-t} \|x\|_{k,p} \|y\|_{k,p}\,, \tag{3.8}$$

$x, y \in \ell^p(\{a_{m,k}\})$, $1 < p < \infty$, $0 < t < 1$.

If we consider $x \in \ell_i^p(\{a_{m,k}\})$ and $y \in \ell_n^p(\{a_{m,k}\})$, $n > i$, then

$$\|B(x,y)\|_{k,p} \leq C_n^t K_n^{1-t} \|x\|_{k,p} \|y\|_{k,p}\,.$$

If $x \in \lambda_i^p(A)$, let be $n \in \mathbb{N}$, $n > i$, such that $C_n^t K_n^{t-1} \|x\|_{k,p} < 1$. For any $k \in \mathbb{N}$, if $y \in \ell_n^p(\{a_{m,k}\})$, by (3.8) we have

$$\|\Gamma_i(x)y\|_{k,p} = \|B(x,y)\|_{k,p} \leq C_n^t K_n^{t-1} \|x\|_{k,p} \|y\|_{k,p} < \|y\|_{k,p}\,,$$

that is, the operator $\Gamma_i(x)\colon \ell_n^p(\{a_{m,k}\}) \longrightarrow \ell_n^p(\{a_{m,k}\})$ is continuous with $\|\Gamma_i(x)\| < 1$. Then, $V(x) = I + \Gamma_i(x)$ is an isomorphism from $\ell_n^p(\{a_{m,k}\})$ to itself, for all k, and the Theorem follows. □

Remark (3.9). In analytic spaces the previous results read:

1. Assume $\lambda^1(A) = \lambda^0(A) = \mathcal{H}(\mathbb{D})$. Then:

a. If T is a basis operator, it follows that $\forall\, k \in \mathbb{N}$

$$\sup_{\substack{j\geq 1\\ q\geq i+1}} \left\{ \frac{\lambda_{j+i}\cdots\lambda_{j+q-1}}{\lambda_i\cdots\lambda_{q-1}} e^{-\frac{q+j}{k}} \right\} < \infty,$$

for all $i = 0, 1, 2, \ldots$

b. If $\lim\limits_{n\to\infty} \lambda_n = 0$ and

$$\sup_{\substack{j\geq 1\\ q\geq i+1}} \left\{ \frac{\lambda_{j+i}\cdots\lambda_{j+q-1}}{\lambda_i\cdots\lambda_{q-1}} \right\} < \infty,$$

for all $i = 0, 1, 2, \ldots$, then T is a basis operator.

2. Assume $\lambda^1(A) = \lambda^0(A) = \mathcal{H}(\mathbb{C})$. Then:

a. If T is a basis operator, it follows that $\exists\, r, s \in \mathbb{N}$ and such that

$$\sup_{\substack{j\geq 1\\ q\geq i+1}} \left\{ \frac{\lambda_{j+i}\cdots\lambda_{j+q-1}}{\lambda_i\cdots\lambda_{q-1}} e^{-qr} e^{-(i+j)s} \right\} < \infty,$$

for all $i = 0, 1, 2, \ldots$

b. If $\lim_{n\to\infty} \lambda_n = 0$ and

$$\sup_{\substack{j\geq 1\\ q\geq i+1}} \left\{ \frac{\lambda_{j+i}\cdots\lambda_{j+q-1}}{\lambda_i\cdots\lambda_{q-1}} \right\} < \infty,$$

for all $i = 0, 1, 2, \ldots$, then T is a basis operator.

Therefore the operator T given by $\lambda_n = \frac{1}{n+1}$ (integration operator) is a basis operator on both spaces $\mathcal{H}(\mathbb{D})$ and $\mathcal{H}(\mathbb{C})$ as

$$\sup_{\substack{j\geq 1\\ q\geq i+1}} \left\{ \frac{\frac{1}{i+j+1}\cdots\frac{1}{j+q}}{\frac{1}{i+1}\cdots\frac{1}{q}} \right\} \leq 1.$$

and so the system $\{n!T^n x\}$, $x_0 \neq 0$ is a basis.

As corollaries of the Theorems of Section (3), we can obtain an extension of Theorem 1 and Theorem 2 of [14] for weighted ℓ^p an c_0 spaces.

Proposition (3.10). *Let T be a weighted shift operator from $c_0(\{a_{n,r}\})$ to $c_0(\{a_{n,k}\})$.*

1. *If T is a basis operator, then the following condition is true:*

$$\forall\, i \geq 0, \qquad \sup_{m\geq i+2} \sum_{s=0}^{m-i-2} \frac{\lambda_{m-s-1}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{i+s}} \frac{a_{m,k}}{a_{m-s-1,r}a_{s+i+1,r}} < \infty$$

2. *Suppose that*

$$\lim_{n\to\infty} \lambda_n \frac{a_{n+1,k}}{a_{n,r}} = 0$$

and $a_{n,r} \leq C\, a_{n,k}$. Then the condition stated in part 1 implies

$$\lim_{n\to\infty} \left(\sup_{m\geq n} \sum_{s=0}^{m-i-2} \frac{\lambda_{m-s-1}\cdots\lambda_{m-1}}{\lambda_i\cdots\lambda_{i+s}} \frac{a_{m,k}}{a_{m-s-1,r}a_{s+i+1,r}} \right) = 0,\ \forall\, i = 0, 1, 2, \ldots$$

3. *Assuming the first condition of part 2 and $k = r$, then the inequality stated in part 1 is also sufficient for the operator T to be a basis operator.*

For weighted ℓ^1 spaces we have

Proposition (3.11). *Let T be a weighted shift operator from $\ell^1(\{a_{n,r}\})$ to $\ell^1(\{a_{n,k}\})$.*

1. *If T is a basis operator, then*

$$\forall\, i \geq 0, \qquad \sup_{s\geq 1, m\geq 0} \left\{ \frac{\lambda_{s+i}\cdots\lambda_{s+i+m}}{\lambda_i\cdots\lambda_{i+m}} \frac{a_{s+i+m+1,k}}{a_{i+s,r}a_{i+m+1,r}} \right\} < \infty.$$

2. *The conditions*

$$\forall\, i \geq 0, \qquad \sup_{s\geq 1, m\geq 0} \left\{ \frac{\lambda_{s+i}\cdots\lambda_{s+i+m}}{\lambda_i\cdots\lambda_{i+m}} \frac{a_{s+i+m+1,k}}{a_{i+s,k}a_{i+m+1,k}} \right\} < \infty.$$

and

$$\lim_{n\to\infty} \lambda_n \frac{a_{n+1,k}}{a_{n,k}} = 0.$$

imply that the operator T is a basis operator when $r = k$.

Remark (3.12). Note that taking $\lim_{n\to\infty} (\lambda_0\lambda_1 \cdots \lambda_n a_{n,k})^{\frac{1}{n}} = 0$ instead of $\lim_{n\to\infty} \lambda_n \frac{a_{n+1,k}}{a_{n,k}} = 0$ the previous result is still true. The proof is similar to Theorem 2 of [14].

PROPOSITION (3.13). *Suppose that*

$$\lim_{n\to\infty} \lambda_n \frac{a_{n+1,k}}{a_{n,k}} = 0.$$

Then the condition

$$\forall\, i \geq 0, \qquad \sup_{m\geq i+2} \sum_{s=0}^{m-i-2} \frac{\lambda_{m-s-1}\cdots\lambda_{m-1}}{\lambda_i \cdots \lambda_{i+s}} \frac{a_{m,k}}{a_{m-s-1,k}a_{s+i+1,k}} < \infty$$

is necessary and sufficient for the operator T to be a basis operator on $\ell^p(\{a_{n,k}\})$, $p \in [1,\infty)$ (for $p = \infty$ take c_0).

Remark (3.14). Note that the condition $\liminf |\lambda_0 \ldots \lambda_n|^{1/n} = 0$ or equivalently $\lim (\lambda_0 \ldots \lambda_n a_{n,k})^{1/n} = 0$ may certainly not hold for several important shifts. It would be nice to have some examples of those, as well as "some direct converses" of the necessary conditions obtained in Theorems (3.1), (3.3), (3.5). These problems open an interesting line of research that we intend to pursue in the near future.

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DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE SALAMANCA
PLAZA DE LA MERCED 1-4
E-37008 SALAMANCA
SPAIN
cordero@usal.es
prada@usal.es

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ON WALLMAN BASES AND COMPACTIFICATIONS

A. GARCÍA-MÁYNEZ C.

ABSTRACT. We consider three types of compactifications of a T_1-topological space X: *a*) perfect ones; *b*) compactifications in which X is locally connected and *c*) Hausdorff compactifications of X (in case X is completely regular) in which X is G_δ-dense. We give necessary and sufficient conditions on a Wallman basis $\mathcal{B}$ of X for the induced compactification to be of type a) or b) (when $\mathcal{B}$ is not necessarily normal) and characterize compactifications of type c) in terms of the existence of a normal Wallman basis on X which satisfies a certain condition.

1. Introduction

Wallman bases exist in every T_1-topological space X and each one of them induces a compactification of X. The old problem as to whether every Hausdorff compactification of a Tychonoff space is induced by a normal Wallman basis of X was solved in the negative by V.M. Uljanov [U]. However, if we require the compactification to satisfy an extra condition, the answer may be affirmative. For instance, if Z is a Hausdorff compactification in which X is G_δ-dense, then Z is induced by a certain normal Wallman basis of X (See (4.3) below). We do not know if every perfect or locally connected compactification of X is of Wallman type. However, if we know that the compactification Z is induced by a Wallman basis $\mathcal{B}$ of X, we describe necessary and sufficient conditions on $\mathcal{B}$ for Z to be perfect and for X to be locally connected in Z. These problems have already been solved when Z is Hausdorff (See [G1] and [G2]).

2. Definitions and preliminary results

Definitions (2.1). a) If X is a set and if $\mathcal{G} \subseteq \mathcal{P}(X)$, $C(\mathcal{G})$ denotes the family of complements in X of elements of $\mathcal{G}$, *i.e.*,

$$C(\mathcal{G}) = \{X - G \mid G \in \mathcal{G}\}.$$

b) Let X be a topological space and let $\mathcal{B}$ be a basis for the open sets of X. We say $\mathcal{B}$ is a *ring basis* of X if $B, B' \in \mathcal{B}$ implies $B \cap B' \in \mathcal{B}$ and $B \cup B' \in \mathcal{B}$.

c) A *Wallman basis* for a topological space X is a ring basis $\mathcal{B}$ of X which satisfies the additional condition:

$*$) If $x \in B \in \mathcal{B}$, there exists an element $H_x \in C(\mathcal{B})$ such that $x \in H_x \subseteq B$.

d) A topological space (X, τ) is R_0 if τ is a Wallman basis of X, *i.e.*, if every open set is a union of closed sets.

e) A Wallman basis $\mathcal{B}$ for a topological space (X, τ) is said to be *normal* if every two disjoint members of $C(\mathcal{B})$ are contained in disjoint members of $\mathcal{B}$.

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f) A non-empty subfamily ξ of $C(\mathcal{B})$ is a *Wallman ultrafilter* if ξ satisfies the following conditions:

1) Each element $H \in \xi$ is non-empty.
2) For every pair of elements $H, K \in \xi$, $H \cap K$ also belongs to ξ.
3) If $H \in \xi$ and $H \subseteq K \in C(\mathcal{B})$, then $K \in \xi$.
4) An element $K \in C(\mathcal{B})$ belongs to ξ if and only if $K \cap H \neq \emptyset$ for every $H \in \xi$.

Observe that for a Wallman basis $\mathcal{B}$ of X, every point $p \in X$ determines a Wallman ultrafilter, namely $\xi_p = \{H \in C(\mathcal{B}) \mid p \in H\}$. The collection of all Wallman ultrafilters is denoted as $X(\mathcal{B})$. There is a natural map $v\colon X \to X(\mathcal{B})$ which assigns to every $p \in X$ its fixed ultrafilter ξ_p. For every $A \subseteq X$ we define a subset A^* of $X(\mathcal{B})$ by means of the formula

$$A^* = \{\xi \in X(\mathcal{B}) \mid \text{ for some } F \in \xi, F \subseteq A\}.$$

Clearly $A \subseteq C \subseteq X$ implies $A^* \subseteq C^* \subseteq X^* = X(\mathcal{B})$ and for every pair of subsets C, D of X we have:

$$\left(C \cap D\right)^* = C^* \cap D^*.$$

The formula $\left(C \cup D\right)^* = C^* \cup D^*$ also holds provided that C and D both belong to $\mathcal{B} \cup C(\mathcal{B})$ (see [GT]). The family $\mathcal{B}^* = \left\{B^* \mid B \in \mathcal{B}\right\}$ is a basis for a compact topology $\tau_{\mathcal{B}}$ of $X(\mathcal{B})$. The natural mapping $v : (X, \tau) \to (X(\mathcal{B}), \tau_{\mathcal{B}})$ is then continuous, open onto its range, with fibers $\overset{-1}{v}v(p) = \{p\}^-$ and its range $v(X)$ is dense in $X(\mathcal{B})$. Therefore, if (X, τ) is a T_1-space, the pair $(v, X(\mathcal{B}))$ is a T_1-compactification of X, called the *associated Wallman compactification* of X with respect to the basis $\mathcal{B}$.

The following results are well known:

(2.2). *A topological space is R_0 if and only if the closures of two points are either the same or disjoint.*

(2.3). *A topological space is completely regular if and only if its topology admits a normal Wallman basis.*

(2.4). *The Wallman compactification $X(\mathcal{B})$ of a T_1-space X with Wallman basis $\mathcal{B}$ is a Hausdorff space if and only if $\mathcal{B}$ is a normal basis.*

(2.5). *If X is a completely regular Hausdorff space (i.e., a Tychonoff space), then the family of cozero sets in X is a normal Wallman basis of X whose induced compactification is homeomorphic to the Stone-Čech compactification βX.*

(2.6). *If $\mathcal{B}$ is a normal Wallman basis of a Tychonoff space (X, τ), then the family $U(\mathcal{B})$ of finite covers of X with elements of $\mathcal{B}$ is a compatible uniformity basis of (X, τ) and the uniform completion $(X, U(\mathcal{B}))\hat{}$ is a compactification of X homeomorphic to the Wallman compactification $(v, X(\mathcal{B}))$ (see* [GT]*, 4.46).*

We recall now some definitions.

(2.7). a) *A space Z is an extension of a space X if X is a dense subspace of Z. A compact extension of X is called a compactification of Z.*

b) *Two extensions Z_1, Z_2 of a space X are equivalent if there exists an homeomorphism $\varphi : Z_1 \to Z_2$ such that $\varphi|_X$ is the identity map of X.*

c) *Let H, A, B be subsets of a space X. We say that H separates A, B in X if there exist two subsets $L, M \subseteq X$ such that $X - H = L \cup M$, $A \subseteq L$, $B \subseteq M$ and $L \cap C\ell_X(M) = \emptyset = \left[C\ell_X(L)\right] \cap M$.*

d) *An extension Z of X is perfect if whenever a closed subset H of X separates two sets $A, B \subseteq X$ in X, the set $C\ell_Z(H)$ (i.e., the closure of H in Z) separates A, B in Z.*

e) *Let Z be an extension of X. For every set $A \subseteq X$, define a subset $A_1 \subseteq Z$ by means of the formula:*

$$A_1 = Z - C\ell_Z(X - A).$$

f) *X is locally connected in one of its extensions Z if Z has a basis $\{W_i \mid i \in J\}$ such that each restriction $W_i \cap X$ is a connected subset of X. Each W_i is then connected and Z is locally connected.*

THEOREM (2.8). *Consider the operator $A \mapsto A_1$ defined in (2.7) e). Then:*

i) *For every $A \subseteq X$, $A_1 \cap X$ is the interior of A with respect to X. Also, $A \subseteq B$ implies $A_1 \subseteq B_1$.*

ii) *If A is open in X, W is open in Z and if $W \cap X = A$, then $W \subseteq A_1$.*

iii) *For every pair of subsets $A, B \subseteq X$, we have $(A \cap B)_1 = A_1 \cap B_1$.*

iv) *Z is a perfect extension of X if and only if for every pair of disjoint open subsets A, B of X, we have $(A \cup B)_1 = A_1 \cup B_1$.*

Proof. We prove only *iv*). Suppose Z is a perfect extension of X. By *i*), we have only to prove that $(A \cup B)_1 \subseteq A_1 \cup B_1$. Let $z \in (A \cup B)_1$. Consider the closed subset of X:

$$K = X - (A \cup B).$$

By hypothesis, K separates A and B in X. Since Z is a perfect extension of X, $C\ell_Z(K)$ separates A and B in Z. Hence there exist two disjoint open sets U, V in Z such that $U \supseteq A, V \supseteq B$ and $Z - C\ell_Z(K) = U \cup V$. But $z \in (A \cup B)_1 = Z - C\ell_Z(K) = U \cup V$. Then $z \in U$ or $z \in V$. If $z \in U$, then $z \in A_1$ because $U \cap (B \cup K) = \emptyset$ and hence $z \notin C\ell_Z\left(X - A\right)$. Similarly, if $z \in V$, we have $z \in B_1$. Therefore $(A \cup B)_1 \subseteq A_1 \cup B_1$ and the proof of this part is complete.

Assume now that $(A \cup B)_1 = A_1 \cup B_1$ whenever A and B are disjoint open subsets of X. Let K be a closed subset of X separating two sets $C, D \subseteq X$. Then there exist disjoint open sets A, B in X such that $C \subseteq A$, $D \subseteq B$ and $X - K = A \cup B$. We have $C\ell_Z(K) \cap (A_1 \cup B_1) = \emptyset$. Indeed, if $z \in C\ell_Z(K)$, then $z \notin (A \cup B)_1 = A_1 \cup B_1$ and hence $z \notin A_1 \cup B_1$. But on the other hand, the fact that $X = K \cup A \cup B$ implies that $Z = \left[C\ell_Z(K)\right] \cup (A \cup B)_1 = \left[C\ell_Z(K)\right] \cup A_1 \cup B_1$. Hence $C\ell_Z(K)$ separates C and D in Z and the proof is complete. □

3. Main results

We first characterize perfect Wallman compactifications.

THEOREM (3.1). *Let $\mathcal{B}$ be a Wallman basis of a T_1-space (X, τ). Then the following conditions are equivalent:*

i) *$X(\mathcal{B})$ is a perfect extension of $v(X)$.*

ii) *If $K \subseteq B$, where $K \in C(\mathcal{B})$ and $B \in \mathcal{B}$ and if L is an open set in X such that $B \cap Fr(L) = \emptyset$, then there exists a set $B_L \in \mathcal{B}$ such that $K \cap L \subseteq B_L \subseteq B \cap L$.*

iii) *If $K \subseteq L \cup M = B$, where $K \in C(\mathcal{B})$, $B \in \mathcal{B}$ and L, M are disjoint open sets of X, then there exist basic sets $B_L, B_M \in \mathcal{B}$ such that $K \cap L \subseteq B_L \subseteq L$ and $K \cap M \subseteq B_M \subseteq M$.*

Proof. *i)* $\Rightarrow$ *ii)* Let K, L, B be as in condition *ii)*. Define $C = X - B$ and $M = X - C\ell(L)$. Obviously C is closed in X and C separates $B \cap L$ and $B \cap M$ in X. Since $X(\mathcal{B})$ is a perfect extension of $v(X)$, there exist two disjoint open sets L_0, M_0 in $X(\mathcal{B})$ such that $L_0 \cap v(X) = v(B \cap L)$, $M_0 \cap v(X) = v(B \cap M)$ and $X(\mathcal{B}) - C\ell_{X(\mathcal{B})}\big(v(C)\big) = L_0 \cup M_0$. From Theorem (2.8), part *iv)*, we know that

$$X(\mathcal{B}) - C\ell_{X(\mathcal{B})}\big(v(C)\big) = (B \cap L)_1 \cup (B \cap M)_1 .$$

But since $L_0 \cap v(X) = v(B \cap L) = (B \cap L)_1 \cap v(X)$ and $M_0 \cap v(X) = v(B \cap M) = (B \cap M)_1 \cap v(X)$, we must have $L_0 = (B \cap L)_1$ and $M_0 = (B \cap M)_1$. Since $(B \cap L)_1 \cap C\ell_{X(\mathcal{B})}\big(v(K)\big) = (B \cap L)_1 \cap K^*$ is closed in $X(\mathcal{B})$ (because $K^* \subset B^* = B_1 = (B \cap L)_1 \cup (B \cap M)_1$) and since $X(\mathcal{B})$ is compact, there exists a set $B_L \in \mathcal{B}$ such that:

$$(B \cap L)_1 \cap K^* \subseteq B_L^* \subseteq (B \cap L)_1 .$$

Intersecting with $v(X)$ and applying $\overset{-1}{v}$, we have:

$$K \cap L \subseteq B_L \subset B \cap L ,$$

as required.

ii) $\Rightarrow$ *iii)* (Obvious).

iii) $\Rightarrow$ *i)* Let L, M be disjoint open sets in X. By Remark 2.5 *iv)*, it is enough to prove that $(L \cup M)_1 \subseteq L_1 \cup M_1$. Let $\xi \in (L \cup M)_1$. Hence, there exists a basic set $B \in \mathcal{B}$ such that $\xi \in B^* \subseteq (L \cup M)_1$. Let $F \in \xi$ be such that $F \subseteq B$. By hypothesis, there exist basic sets B_L, B_M such that $F \cap L \subseteq B_L \subseteq B \cap L$ and $F \cap M \subseteq B_M \subseteq B \cap M$. Then $\xi \in (B_L \cup B_M)^* = B_L^* \cup B_M^* = (B_L)_1 \cup (B_M)_1 \subseteq L_1 \cup M_1$ and the proof is complete. □

COROLLARY (3.2). *Let (X, τ) be a T_1-space. Then $X(\tau)$ is perfect compactification of X.*

We now obtain a better result, but we need a definition:

Definition (3.3). A Wallman basis $\mathcal{B}$ on a space X is said to be *fine* if whenever $C \subseteq B \in \mathcal{B}$, where C is open and closed with respect to B, we have $C \in \mathcal{B}$.

THEOREM (3.4). *Let $\mathcal{B}$ be a fine Wallman basis on a T_1-space X. Then $X(\mathcal{B})$ is a perfect compactification of X (we have identified X with $v(X)$).*

Proof. It is enough to take $B_L = B \cap L$ in condition *ii)*. □

Since the family of cozero sets in a Tychonoff space X is a fine normal Wallman basis on X, we deduce that βX is a perfect Hausdorff compactification of X, a well known fact in general topology.

We now set ourselves the following problem:

Let $\mathcal{B}$ be a Wallman basis of a T_1-space (X, τ). Find necessary and sufficient conditions on $\mathcal{B}$ which insure that $v(X)$ is locally connected in $X(\mathcal{B})$.

We need some definitions.

Definition (3.5). a) A Wallman basis $\mathcal{B}$ for a space (X,τ) is *locally connected* if for every $B\in\mathcal{B}$ and every component C of B, we have $C\in\mathcal{B}$.

b) A subspace L of a topological space X is said to be *insular* if L has only a finite number of (connected) components.

c) A Wallman basis $\mathcal{B}$ for a space (X,τ) is *pre-fine* if whenever $C\subseteq B$, where $C\in C(\mathcal{B})$ and $B\in\mathcal{B}$, there exists an insular set L such that $C\subseteq L\subseteq B$.

For example, the topology τ of a T_1 locally connected space (X,τ) is a locally connected Wallman basis for (X,τ). However, τ is pre-fine only if every closed discrete subset of X is finite; i.e., only if X is countably compact.

We observe also that if every element of a Wallman basis $\mathcal{B}$ of an insular space (X,τ) has compact boundary, then $\mathcal{B}$ is pre-fine.

It is also easy to prove, using Theorem (3.1), that every locally connected pre-fine Wallman basis is perfect.

We may now prove the following:

THEOREM (3.6). *Let $\mathcal{B}$ be a locally connected Wallman basis of a T_1-space (X,τ). Then $v(X)$ is locally connected in $X(\mathcal{B})$ if and only if $\mathcal{B}$ is pre-fine.*

Proof. (*Sufficiency*). It is enough to prove that $\{B^*\mid B\in\mathcal{B}, B \text{ connected}\}$ is a basis for $X(\mathcal{B})$. (Recall $B^*\cap v(X)=v(B)$ for every $B\in\mathcal{B}$). Suppose that $\xi\in D^*$, where $D\in\mathcal{B}$. There exists then an element $H\in\xi$ such that $H\subseteq D$. Since $\mathcal{B}$ is pre-fine, there exists a finite collection $\{D_1,D_2,\dots,D_n\}$ of components of D covering H. Therefore, $\xi\in(\bigcup_{i=1}^{n} D_i)^*=\bigcup_{i=1}^{n} D_i^*$ and, for some $i_0\in\{1,2,\dots,n\}$, we have $\xi\in D_{i_0}^*\subseteq D^*$.

(*Necessity*). Proceeding by contradiction, suppose there exist elements $H\in C(\mathcal{B})$, $B\in\mathcal{B}$, with $H\subseteq B$ and an infinite collection $\{B_i\mid i\in J\}$ of components of B intersecting H and such that $H\subseteq\bigcup\{B_i\mid i\in J\}$. For each finite set $L\subseteq J$, the set $K_L=H-\bigcup_{i\in L} B_i$ is non-empty and belongs to $C(\mathcal{B})$. Besides, if $L,L'\subseteq J$ are both finite, then:

$$K_{L\cup L'}=K_L\cap K_{L'}\,.$$

Therefore, using Zorn's Lemma, we may find a Wallman ultrafilter $\xi\in X(\mathcal{B})$ such that $K_L\in\xi$ for each $L\subseteq J$ finite. Therefore, ξ must belong to B^*. Let $\{W_s\mid s\in S\}$ be a basis for the open sets in $X(\mathcal{B})$ such that each intersection $W_s\cap v(X)$ is a region in $v(X)$ (this basis exists by hypothesis). Let $s\in S$ be such that $\xi\in W_s\subseteq B^*$. Since $v^{-1}(W_s)$ is connected and is contained in B, there exists a component E of B such that $v^{-1}(W_s)\subseteq E$. Let $D\in\mathcal{B}$ be such that $\xi\in D^*\subseteq W_s$ and let $H'\in\xi$ be such that $H'\subseteq D$. Therefore

$$H'\subseteq D=v^{-1}(D^*)\subseteq v^{-1}(W_s)\subseteq E\,.$$

However, $E=B_{i_0}$ for some $i_0\in J$, because $E\cap H\supset H'\cap H\neq\emptyset$. Therefore, $K_{\{i_0\}}=H-E$ belongs also to ξ and is disjoint from H'. This contradiction proves that the family of components of B intersecting H is finite. □

This theorem has an interesting corollary:

COROLLARY (3.7). *Let $\mathcal{B}$ be a normal and locally connected Wallman basis of a Tychonoff space (X, τ). Let $\mathcal{U}, \mathcal{U}'$ be the collections of finite covers of X consisting of elements of $\mathcal{B}$ (respectively, of connected elements of $\mathcal{B}$). Then $\mathcal{U}$ and $\mathcal{U}'$ generate the same uniformity if and only if $\mathcal{B}$ is pre-fine.*

Proof. Suppose $\mathcal{B}$ is pre-fine. We have only to prove that every cover $\alpha \in \mathcal{U}$ is refined by a cover $\alpha' \in \mathcal{U}'$. Suppose $\alpha = \{B_1, B_2, \ldots, B_n\}$, where each $B_i \in \mathcal{B}$. Since $\mathcal{B}$ is normal, there exist co-basic sets $H_1, H_2, \ldots, H_n \in C(\mathcal{B})$ such that $X = H_1 \cup H_2 \cup \cdots \cup H_n$ and $H_i \subseteq B_i$ for each $i = 1, 2, \ldots, n$. Since $\mathcal{B}$ is pre-fine, there exist insular sets $D_1, \ldots, D_n \in \mathcal{B}$ such that $H_i \subseteq D_i \subseteq B_i$. Hence, the family of components of the D_i's constitute a cover $\alpha' \in \mathcal{U}'$ which refines α.

Suppose now that $\mathcal{U}$ and $\mathcal{U}'$ generate the same uniformity. Let $K \in C(\mathcal{B})$, $B \in \mathcal{B}$ be such that $K \subseteq B$. Since $\alpha = \{B, X - K\} \in \mathcal{U}$, there exists a cover $\alpha' = \{E_1, E_2, \ldots, E_n\}$ in $\mathcal{U}'$ which refines α. With no loss of generality, we may suppose that $E_1, E_2, \ldots, E_s$ is the complete list of elements of α' which intersect K. Then $K \subseteq \bigcup_{j=1}^{s} E_j \subseteq B$ and $\mathcal{B}$ is pre-fine. □

It is now our purpose to find conditions which insure that certain Wallman bases are pre-fine. We need some definitions:

Definition (3.8). Let (X, τ) be a topological space, let $A \subseteq X$ and $\mathcal{B} \subseteq \tau$.

1) We say that A is *C-bounded* with respect to $\mathcal{B}$ if for every discrete family $\{B_i \mid i \in J\}$ consisting of elements of $\mathcal{B}$, the set $\{i \in J \mid A \cap B_i \neq \emptyset\}$ is finite.

2) Suppose $A \subseteq B \subseteq X$. We say that A is *R-embedded* in B if there exists an insular set K such that $A \subseteq K \subseteq B$.

3) A is *R-bounded* with respect to $\mathcal{B}$ if for every discrete family $\{B_i \mid i \in J\}$, consisting of elements of $\mathcal{B}$ and covering A, there exists a finite set $J_0 \subseteq J$ such that $A \subseteq \bigcup\{B_i \mid i \in J_0\}$.

4) A is *C-discrete* respect to $\mathcal{B}$ if there exists a discrete family $\{B_i \mid i \in J\} \subseteq \mathcal{B}$ covering A such that each B_i contains exactly one point of A.

5) If (X, τ) is completely regular, we say that A is *C-bounded* (respectively, *R-bounded*, *C-discrete*) in X if A is C-bounded (respectively, R-bounded, C-discrete) with respect to the family of cozero sets of (X, τ).

The following results are now obvious:

Remarks (3.9). a) Every set which is C-bounded with respect to $\mathcal{B}$ is also R-bounded with respect to $\mathcal{B}$.

b) Every insular set $A \subseteq X$ is R-embedded in every set in which it is contained and it is R-bounded with respect to any family $\mathcal{B} \subseteq \tau$.

c) Every compact set $A \subseteq X$ is R-bounded with respect to any family $\mathcal{B} \subseteq \tau$ and it is R-embedded in every set with open components which contains it.

We prove now the following:

LEMMA (3.10). *Let $\mathcal{B}$ be a locally connected Wallman basis in an insular space (X, τ) and let $H \in C(\mathcal{B})$, $B \in \mathcal{B}$ be such that $H \subseteq B$.*

a) *If $Fr\,(H)$ is R-embedded in B, then H is R-embedded in B.*

b) *If* $\mathcal{B}$ *is normal and* $Fr(H)$ *is R-bounded with respect to* $\mathcal{B}$*, then* H *is R-embedded in* B.

Proof. a) Let $\{B_i \mid i \in J\}$ be the components of B intersecting H and let $J_0 = \{i \in J \mid B_i \cap Fr(H) \neq \emptyset\}$. Any component B_i, with $i \in J - J_0$, is open and closed in X, because $Fr(B_i) \subseteq Fr(B)$, by local connectedness, and $Fr(B_i) \subseteq H$ because $B_i \cap H \neq \emptyset$, $B_i \cap Fr(H) = \emptyset$ imply that $B_i \subseteq H$. Since X is insular, the set $J - J_0$ is finite. But by hypothesis, the set J_0 is also finite. Hence J is finite and H is R-embedded in B.

b) Since $\mathcal{B}$ is normal, there exist sets $E \in \mathcal{B}$ and $K \in C(\mathcal{B})$ such that $H \subseteq E \subseteq K \subseteq B$. The family $\{E \cap B_i \mid i \in J\}$ is discrete in X: to prove this, take any point $x \in X$. If $x \in B$ and B_{i_0} is the component of B containing x, B_{i_0} is a neighborhood of x which intersects, at most, one element of the family $\{E \cap B_i \mid i \in J\}$. If $x \notin B$, $X - K$ is a neighborhood of x which intersects no member of the family $\{E \cap B_i \mid i \in J\}$. Since $Fr(H)$ is R-bounded with respect to $\mathcal{B}$, there exist indices $i_1, i_2, \ldots, i_n \in J$ such that $Fr(H) \subseteq \bigcup_{k=1}^{n} (E \cap B_{i_k})$ and hence the set J_0 is finite. We already know that $J - J_0$ is finite. Hence J is finite and H is R-embedded in B. □

4. Applications

The final problem we attack is the following:

Let $\mathcal{B}$ be a normal Wallman basis of a Tychonoff space (X, τ). Under what conditions is $v(X)$ G_δ-dense in $X(\mathcal{B})$?

Recall a space X is G_δ-dense in an extension Z if every non-empty G_δ subset of Z intersects X.

We define now some new types of Wallman bases:

Definition (4.1). Let $\mathcal{B}$ be a normal Wallman basis of a Tychonoff space (X, τ).

1) $\mathcal{B}$ is *countably paracompact* if for every countable cover $\{B_1, B_2, \ldots\}$ of X with elements of $\mathcal{B}$, there exists a sequence $\{H_1, H_2, \ldots\}$ of elements of $C(\mathcal{B})$ such that $H_n \subseteq B_n$ for every $n \in \mathbb{N}$ and such that $X = \bigcup_{n=1}^{\infty} H_n$.

2) $\mathcal{B}$ is *special* if for every discrete family $\{B_i \mid i \in J\}$ with respect to X and consisting of elements of $\mathcal{B}$ and for every choice $\{H_i \mid i \in J\}$ of sets in $C(\mathcal{B})$ with $H_i \subseteq B_i$ for every $i \in J$, we have $\bigcup_{i \in J} H_i \in C(\mathcal{B})$ and $\bigcup_{i \in J} B_i \in \mathcal{B}$.

3) $\mathcal{B}$ is *countably compact* if every countable cover of X with elements of $\mathcal{B}$ has a finite subcover.

4) A decreasing sequence $B_1 \supseteq B_2 \supseteq \ldots$ of non-empty elements of $\mathcal{B}$ is a *regular Wallman sequence* if for every n there exists a cobasic set $H_n \in C(\mathcal{B})$ such that $B_{n+1} \subseteq H_n \subseteq B_n$.

We prove the following results:

THEOREM (4.2). a) *Every countably compact Wallman basis is countably paracompact.*

b) *Every countably compact, locally connected, special Wallman basis is prefine.*

c) *Every prefine, countably paracompact, special Wallman basis is countably compact.*

Proof. a) (Obvious).

b) Let $H \in C(\mathcal{B})$ and $B \in \mathcal{B}$ be such that $H \subseteq B$. Let $D \in \mathcal{B}$ and $K \in C(\mathcal{B})$ be such that $H \subseteq D \subseteq K \subseteq B$. Let $\{B_i \mid i \in J\}$ be the components of B intersecting H. As in 3.10 *b*), we prove that the family $\{D \cap B_i \mid i \in J\}$ is discrete with respect to X. In fact, this family is finite: assuming the contrary, select different indices $i_1, i_2, \dots \in J$ and choose points x_k and cobasic sets H_k such that $x_k \in H_k \subseteq D \cap B_{i_k}$. By hypothesis, $H_0 = \bigcup_{k=1}^{\infty} H_k \in C(\mathcal{B})$. Also, by hypothesis, the countable cover

$$\alpha = \{D \cap B_{i_1}, D \cap B_{i_2}, \dots\} \cup \{X - H_0\}$$

has a finite subcover. But this contradicts the fact that α is an irreducible cover of X. Therefore, J is finite and $L = \bigcup\{B_i \mid i \in J\}$ is an insular set such that $H \subseteq L \subseteq B$.

c) Let $\alpha = \{B_1, B_2, \dots\}$ be a countable cover of X by elements of $\mathcal{B}$. Suppose, contrary to what we want to prove, that $X \neq \bigcup_{n=1}^{s} B_n$ for every $s \in \mathbb{N}$. We may then find an infinite sequence $A = \{x_1, x_2, \dots\}$ such that for each $n \in \mathbb{N}$, $\{k \mid x_k \in B_n\}$ is a finite set. Since by hypothesis the cover α satisfies the conditions in [G3], 2.4, the set A is C-discrete. Hence, we may find basic sets $D_1, D_2, \dots$ and cobasic sets $E_1, E_2, \dots$ such that $x_k \in E_k \subseteq D_k$ for each $k \in \mathbb{N}$ and such that the family $\{D_1, D_2, \dots\}$ is discrete. Since $\mathcal{B}$ is prefine and special, there exists an insular set L such that $\bigcup_{k=1}^{\infty} E_k \subseteq L \subseteq \bigcup_{k=1}^{\infty} D_k$. But then $L \subseteq \bigcup_{k=1}^{s} D_k$ for some $s \in \mathbb{N}$, contradicting the fact that $x_{s+1} \in E_{s+1} \subseteq L - \bigcup_{k=1}^{s} D_k$. Therefore, α has a finite subcover and $\mathcal{B}$ is countably compact. □

We prove now our final result (compare with Theorem 3.4 [GG]):

THEOREM (4.3). *Let Z be a compact Hausdorff extension of a space X. Then the following three conditions are equivalent:*

1) *X is G_δ-dense in Z.*

2) *There exists a countably compact Wallman basis $\mathcal{B}_0$ of X such that Z and $X(\mathcal{B}_0)$ are equivalent compactifications of X.*

3) *Z is a Wallman type compactification of X and for every normal Wallman basis $\mathcal{B}$ of X such that Z and $X(\mathcal{B})$ are equivalent compactifications of X and for every regular Wallman sequence $B_1 \supseteq B_2 \supseteq \dots$ of elements of $\mathcal{B}$, the set $\bigcap_{n=1}^{\infty} B_n$ is non-empty.*

Proof. 1) ⇒ 2). Define $\mathcal{B}_0 = \{G \cap X \mid G \text{ is a cozero set in } Z\}$. Clearly $\mathcal{B}_0$ is a Wallman basis of X. We prove now that $\mathcal{B}_0$ is normal. Let H, K be zero sets in Z such that $(H \cap X) \cap (K \cap X) = \emptyset$. Hence $H \cap K \subseteq Z - X$. But being a zero set of Z, $H \cap K$ is a G_δ subset of Z disjoint from X. Since X is G_δ-dense in Z, we must have $H \cap K = \emptyset$. Hence, there exists disjoint cozero subsets G, G' of Z such that $H \subseteq G$ and $K \subseteq G'$. Therefore, $G \cap X$ and $G' \cap X$ are disjoint elements of $\mathcal{B}_0$ containing $H \cap X$ and $K \cap X$, respectively, and $\mathcal{B}_0$ is normal. Since the only F_δ subset of Z which contains X is Z, [GT], 3.48.7 implies that Z and $X(\mathcal{B}_0)$ are equivalent compactifications of X. We finally prove that $\mathcal{B}_0$ is countably compact. Let $G_1, G_2, \dots$ be a sequence of cozero sets of Z such that $X \subseteq \bigcup_{i=1}^{\infty} G_i$. Since $\bigcup_{i=1}^{\infty} G_i$ is also a cozero set in Z and since X is G_δ-dense in Z, we must have $Z = \bigcup_{i=1}^{\infty} G_i$. The compactness of Z implies the existence of a natural number n such that $Z = \bigcup_{i=1}^{n} G_i$. Hence, $\mathcal{B}_0$ is countably compact.

2) $\Rightarrow$ 3) Let $B_1 \supseteq B_2 \supseteq \dots$ be a regular Wallman sequence (with respect to $\mathcal{B}_0$). Suppose, contrary to what we want to prove, that $\bigcap_{n=1}^{\infty} B_n = \emptyset$. By assumption, there exist cobasic sets $H_1, H_2, \dots \in C(\mathcal{B})$ such that $B_{n+1} \subseteq H_n \subseteq B_n$ for every $n = 1, 2, \dots$. The sets H_n^* and $(X - B_n)^*$ are compact and disjoint for every $n = 1, 2, \dots$. (We have identified Z with $X(\mathcal{B})$). Using 2), we may find disjoint elements D_n, E_n in $\mathcal{B}_0$ such that $H_n \subseteq D_n$ and $X - B_n \subseteq E_n$ (Observe the bases $\mathcal{B}$ and $\mathcal{B}_0$ are equivalent). Since $\mathcal{B}_0$ is countably compact and since $\bigcup_{n=1}^{\infty} E_n \supseteq \bigcup_{n=1}^{\infty} (X - B_n) = X - \bigcap_{n=1}^{\infty} B_n = X - \emptyset = X$, we have $X = E_1 \cup E_2 \cup \dots \cup E_{n_0}$ for some integer n_0. But then $H_{n_0} \subseteq \bigcap_{n=1}^{n_0} D_n \subseteq \bigcap_{n=1}^{n_0} (X - E_n) = X - \bigcup_{n=1}^{n_0} E_n = \emptyset$, a contradiction.

3) $\Rightarrow$ 1) Let $V_1 \supseteq V_2 \supseteq \dots$ be open sets in Z with non-empty intersection. We have to prove that $X \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset$. By hypothesis, Z is a Wallman type compactification of X. Hence, there exists a normal Wallman basis $\mathcal{B}$ of X such that Z and $X(\mathcal{B})$ are equivalent compactifications of X. Select a point $z \in \bigcap_{n=1}^{\infty} V_m$. Let $B_1 \in \mathcal{B}$ be such that

$$z \in B_1^* \subseteq C\ell_Z\left(B_1^*\right) \subseteq V_1\,.$$

Since $Z - V_1$ is compact and since $Z - V_1 \subseteq Z - C\ell_Z\left(B_1^*\right)$, there exists a basic set $D_1 \in \mathcal{B}$ such that $Z - V_1 \subseteq D_1^* \subseteq Z - C\ell_Z\left(B_1^*\right)$. Then $H_1 = X - D_1 \in C(\mathcal{B})$ and $C\ell_Z\left(B_1^*\right) \subseteq Z - D_1^* = H_1^* \subseteq V_1$. Assuming the sets $B_k \in \mathcal{B}$, $H_k \in C(\mathcal{B})$ have already been defined for $k < n$, let $B_n \in \mathcal{B}$ be such that:

$$z \in B_n^* \subseteq C\ell_Z\left(B_n^*\right) \subseteq V_n \cap B_{n-1}^*\,.$$

Using the same argument as before, we may find a basic set $D_n \in \mathcal{B}$ such that $Z - (V_n \cap B_{n-1}^*) \subseteq D_n^* \subseteq Z - C\ell_Z\left(B_n^*\right)$. If we define $H_n = X - D_n$, we have:

$$C\ell_Z\left(B_n^*\right) \subseteq H_n^* \subseteq V_n \cap B_{n-1}^*\,.$$

The sequence $B_1 \supseteq B_2 \supseteq \dots$ is then a regular Wallman sequence and, by hypothesis, there exists a point $x \in \bigcap_{n=1}^{\infty} B_n$. Therefore $x \in X \cap \bigcap_{n=1}^{\infty} V_n$ and the proof is complete. □

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Instituto de Matemáticas
Universidad Nacional Autónoma de México
Area de la Investigación Científica
Circuito Exterior
Ciudad Universitaria
04510 México D.F.
México
agmaynez@matem.unam.mx

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HOMOTOPY PERIODIC SETS OF SELFMAPS OF REAL PROJECTIVE SPACES

JERZY JEZIERSKI

Abstract. We say that a natural number k is the homotopy period of a self-map $f\colon X \to X$ if each map homotopic to f has a periodic point with the minimal period k : we denote the set of these numbers HPer(f). Boju Jiang gave a necessary condition for $k \in \mathrm{HPer}(f)$. Here we show that in the case of compact manifolds of dimension ≥ 3 this condition is also sufficient. We also give a formula for HPer(f) for self-maps of real projective space $\mathbb{R}P^n$.

1. Definitions

Let $f\colon X \to X$ be a self-map and $n \in \mathbb{N}$ a natural number. We denote $\mathrm{Fix}(f) = \{x \in X\colon f(x) = x\}$, the fixed point set $P^n(f) = \mathrm{Fix}(f^n)$, and $P_n(f) = \{x \in X; f^n(x) = x,\ f^k(x) \neq x \text{ for } k < n\}$, the set of points of the pure period n. One of the basic problems in the theory of Dynamical Systems is the existence of periodic points of a prescribed pure period. Boju Jiang [Ji1] gave an algebraic sufficient condition for the existence of periodic points of the given pure periods (existence of essential and irreducible Nielsen classes). In this paper we show that in the case of compact PL manifolds of dimension ≥ 3 this condition is also necessary in the sense that the lack of existence of essential and irreducible Nielsen classes implies a homotopy from f to a map g satisfying $P_n(g) = \emptyset$. We will use this result to describe the set of homotopy periods HPer(f) in the case of self-maps of real projective spaces.

The method of the proof will follow the proof of the Wecken Theorem for periodic points: [Je3] , [Je4]. In fact the proof will be a consequence of several procedures proved in these papers.

We recall briefly some definitions. For the details see [Ji1]. Let $f\colon X \to X$ be a selfmap of a compact ANR. We define the *Nielsen relation* on the fixed point set Fix(f). The points $x, y \in \mathrm{Fix}(f)$ are *Nielsen related* if there is a path ω from x to y such that $f(\omega)$ is homotopic to ω by a homotopy keeping the end points fixed. This relation divides Fix(f) into a finite number of *Nielsen classes*. We denote the quotient set by $\mathcal{N}(f)$. On the other hand we consider the action of the fundamental group $\pi_1(X, x_0)$ on itself defined as follows. Choosing a path γ from x_0 to $f(x_0)$, the action is defined by

$$\alpha \circ \omega = \alpha\omega\gamma f(\alpha^{-1})\gamma^{-1}.$$

The quotient set is called the *set of Reidemeister classes* and denoted $\mathcal{R}(f)$. There is a natural inclusion $i\colon \mathcal{N}(f) \to \mathcal{R}(f)$ defined as follows. For a fixed point $x \in \mathrm{Fix}(f)$ we choose a path η from the base point x_0 to x. Then $\eta f(\eta^{-1})\gamma^{-1}$

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is a path representing an element in $\pi_1(X;x_0)$. For $A \in \mathcal{N}(f)$, the Nielsen class containing x, we define $i(A)$ to be the Reidemeister class represented by $\eta f(\eta^{-1})\gamma^{-1}$; i.e., $i(A) = [\eta f(\eta^{-1})\gamma^{-1}]$.

The natural inclusion $\mathrm{Fix}(f) \subset \mathrm{Fix}(f^k)$ induces the map $\mathcal{N}(f) \to \mathcal{N}(f^k)$. On the other hand we have a map $j\colon \mathcal{R}(f) \to \mathcal{R}(f^k)$ given by

$$j[\alpha] = [\alpha\gamma f(\alpha\gamma)...f^{k-2}(\alpha\gamma)f^{k-1}(\alpha)(\gamma f(\gamma)...f^{k-2}(\gamma))^{-1}].$$

Then we have the commutative diagram (see Chapter 3 in [Ji1])

$$\begin{array}{ccc} \mathcal{N}(f) & \longrightarrow & \mathcal{N}(f^k) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \mathcal{R}(f) & \xrightarrow{\ j\ } & \mathcal{R}(f^k) \end{array}$$

Let us note that the formula $\mathcal{R}(f^k) \ni [\alpha] \to [f\alpha] \in \mathcal{R}(f^k)$ defines a map whose k-iterate is the identity; hence it defines an action of $\mathbb{Z}_k$ on $\mathcal{R}(f^k)$. Its orbits will be called *orbits of Reidemeister classes*. The set of these orbits is denoted $\mathcal{OR}(f^k)$ Let l be a divisor of n and let $j_{kl}\colon \mathcal{R}(f^l) \to \mathcal{R}(f^k)$ denote the induced map of Reidemeister sets. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{R}(f^l) & \xrightarrow{\ j_{kl}\ } & \mathcal{R}(f^k) \\ \big\downarrow & & \big\downarrow \\ \mathcal{R}(f^l) & \xrightarrow{\ j_{kl}\ } & \mathcal{R}(f^k) \end{array}$$

where vertical arrows denote the natural actions of the groups $\mathbb{Z}_l$ and $\mathbb{Z}_k$ respectively. This gives the induced maps between the orbit sets of these actions $\bar{j}_{kl}\colon \mathcal{OR}(f^l) \to \mathcal{OR}(f^k)$. We define the *depth* of an orbit $A \in \mathcal{OR}(f^k)$ as the smallest number l such that A belongs to the image of $\bar{j}_{kl}$. Then we write $d(A) = l$. We say that $A \in \mathcal{OR}(f^l)$ *precedes* $B \in \mathcal{OR}(f^k)$ if l divides k and $j_{kl}(A) = B$.

Definition (1.1). An orbit of Reidemeister classes $A \in \mathcal{OR}(f^k)$ is called *irreducible* iff $d(A) = k$; i.e., A is not preceded by any other orbit of smaller depth. An orbit of Reidemeister classes is called *essential* iff the index of the orbit of Nielsen classes corresponding to it is not zero.

Let us notice that an essential irreducible orbit in $A \in \mathcal{OR}(f^k)$ contains at least k points of pure period k.

2. Procedures

We will consider a self-map $f\colon M \to M$ and a fixed natural number n. We will try to deform this map to get $P_n(f) = \emptyset$. First we are going to make the set of periodic points finite and a PL homeomorphism near each n-periodic point. We start by presenting some "model" maps of $\mathbb{R}^m$.

Lemma (2.1). *For each $k \in \mathbb{Z}$ and $m \geq 2$ there is a map $f\colon \mathbb{R}^m \to \mathbb{R}^m$ satisfying*

1. $\mathrm{Fix}(f) = \{0\}$
2. $\|f(x)\| = \lambda\|x\|$ *for an arbitrarily prescribed* $\lambda > 1$;

3. *$f(\alpha x) = \alpha f(x)$ for any $\alpha > 0$ hence f sends semi-lines starting from* 0 *into semi-lines;*

4. *for any l the number of semi-lines, starting from* 0*, which are sent by f^l into itself is finite;*

5. $\mathrm{ind}(f) = k$.

Proof. The maps are given by the formulae: $f(z,v) = \lambda(\rho_{k'}(z), v)$ where $(z,v) \in \mathbb{C} \times \mathbb{R}^{m-2} = \mathbb{R}^m$, $k \in \mathbb{Z}$, $\lambda > 1$ and $\rho_{k'} : \mathbb{C} \to \mathbb{C}$ is given by (in polar coordinates $r \in [0,\infty)$ and $\phi \in \mathbb{R}$ regarded modulo $\pi\mathbb{Z}$).

$$\rho_{k'}(r,\phi) = \begin{cases} (r, k'\phi) & \text{for } |k'| \geq 2, \\ (r, \phi + \alpha_0) & \text{for } k' = 1, \\ (r, \chi_{-1}(\phi)) & \text{for } k' = -1, \\ (r, \chi_0(\phi)) & \text{for } k' = 0, \end{cases}$$

where $k' = (-1)^m k$ and

1. For $k' = 1$: $\alpha_0 > 0$, denotes an irrational angle; i.e., α_0/π is irrational,
2. For $k' = -1$ we define $\chi_{-1} : [-\pi,\pi] \to [-\pi,\pi]$ as a homeomorphism satisfying $-\chi_{-1}(x) > x$ for $0 < x < \pi$, $\chi_{-1}(x) = -x$ for $-\pi \leq x \leq 0$,
3. For $r = 0$, we define $\chi_0 : [-\pi,\pi] \to [-\pi,\pi]$ by the formula $\chi_0(x) = \chi_{-1}(|x|)$.

For the details see [Je4]. □

We will call such a map f a standard map. By Section 4 in [Je2] we may assume that any selfmap of a manifold is homotopic to a map which is standard near each of its periodic points (in a chart). In fact this can be achieved by combining the Cancelling and the Creating Procedures given in the next section. We will say that S_l^k is satisfied at a periodic point if l is the period of this point and k equals the fixed point index of f^k at this point. This property will allow to control periodic points during deformations of f.

Theorem (2.2). [Je1] *Let $M \subset \mathbb{R}^N$ be a compact PL-submanifold with the metric inherited from a Euclidean metric in $\mathbb{R}^N$. Let $n \in \mathbb{N}$ be a fixed number. Then any continuous map $f : M \to M$ is homotopic to a map g such that* $\mathrm{Fix}\,(g^n)$ *is finite and g is a* PL*-homeomorphism near any point $x \in \mathrm{Fix}\,(g^n)$. Moreover, for any $\epsilon > 0$ we may choose a g satisfying $d(f,g) < \epsilon$.* □

The next Theorem is the main result of [Je3] (compare [Je1]). It allows removing two Nielsen related orbits of opposite indices.

Theorem (2.3) (Canceling Procedure (Theorem 2.5 [Je3])). *Let $f : M \to M$ be a selfmap of a PL-manifold of dimension ≥ 3 with* $\mathrm{Fix}(f^k)$ *finite. Moreover we assume that*

1. *$\{x_0, ..., x_{k-1}\}$, $\{y_0, ..., y_{k-1}\}$ are Nielsen related orbits of length k;*
2. *f is a a PL-homeomorphism near each point from $\{x_0, ..., x_{k-1}, y_0, ..., y_{k-1}\}$;*
3. $\mathrm{ind}(f^k; x_0) + \mathrm{ind}(f^k; y_0) = 0$.

Then there is a homotopy $\{f_t\}$ constant in a neighbourhood of

$$\mathrm{Fix}(f^k) \setminus \{x_0, ..., x_{k-1}, y_0, ..., y_{k-1}\}$$

such that $f_0 = f$ *and*

$$\mathrm{Fix}(f_1^k) = \mathrm{Fix}(f^k) \setminus \{x_0, ..., x_{k-1}, y_0, ..., y_{k-1}\}\,.$$

□

The next Procedure enables one to create a new periodic point of given index r. Since the index is the homotopy invariant the new periodic point is balanced by $|r|$ orbits of points each of index $= -\mathrm{sgn}(r)$.

THEOREM (2.4) (Creating Procedure.(Thm. 5.3) [Je2]). *Let* $\{x_0, ..., x_{k-1}\}$ *be an isolated* k*-orbit of a map* $f\colon M \to M$ *which is a local homeomorphism near each* x_i. *We fix a Euclidean neighbourhood* $U \ni x_0$ *such that* $f^i(\mathrm{cl}\, U) \cap f^j(\mathrm{cl}\, U) = \emptyset$ *and* f *is a homeomorphism on each* $f^i(\mathrm{cl}\, U)$ *for* $0 \le i < j \le k-1$. *Let* $V_0 \subset U$ *be a Euclidean neighbourhood of* x_0 *satisfying* $f^k(\mathrm{cl}\, V_0) \subset U$. *We denote* $V_i = f^i(V_0)$ *and suppose that* $0 \in V_0 \subset U = \mathbb{R}^m$. *We may assume that* $x_0 \ne 0$.

Then there exists a homotopy $\{f_t\}$ *constant outside* V_{k-1} *satisfying* $f_0 = f$, $f_1^k(0) = 0$ *(hence* $0 \in V_0$ *becomes a new periodic point) and* $S_r(f^k)$ *is satisfied at this point for a prescribed number* $r \ne 0$. *Moreover still* $f_1(\mathrm{cl}\, V_{k-1}) \subset U$ *and*

$$\mathrm{Fix}(f_1^k) = \mathrm{Fix}(f^k) \cup \{\text{the orbit of } 0\} \cup$$

$$\cup \{\text{orbits of } |r| \text{ new periodic points in } V_{k-1} \text{ each of index} = -\mathrm{sgn}(r)\}\,.$$

□

The next Procedure makes a non-periodic point a periodic one (of index zero) in an arbitrarily prescribed Nielsen class.

THEOREM (2.5) (Addition Procedure). *Given numbers* $k, n \in \mathbb{N}$ *such that* $k|n$, *a map* $f\colon M \to M$ *such that* $\mathrm{Fix}(f^n)$ *is finite and a point* $x_0 \in M$ *such that the points* $x_0, x_1 = f^1(x_0), ..., x_{2n} = f^{2n}(x_0)$ *are different. Let moreover* $\dim M \ge 3$. *Then there is a homotopy* $\{f_t\}_{0 \le t \le 2}$ *satisfying*

1. $f_0 = f$;
2. $\{f_t\}$ *is constant in a neighbourhood of* $\mathrm{Fix}(f^n)$;
3. $f_2^k(x_0) = x_0$ *and* $f_2^i(x_0) \ne x_0$ *for* $i = 1, ..., k-1$;
4. $\mathrm{Fix}(f_2^n) = \mathrm{Fix}(f^n) \cup \{x_0\}$;
5. f_2 *satisfies* S_0^k *at the point* x_0. □

The last procedure enables one to join a longer orbit to shorter one.

THEOREM (2.6) (Coalescing Procedure ([Je4], [Je2])). *Suppose that* $\mathrm{Fix}(f^k)$ *is finite. Let the orbits* $\{y_0, ..., y_{l-1}\} \in \mathrm{Fix}(f^l), \{x_0, ..., x_{k-1}\} \in \mathrm{Fix}(f^k)$ *be disjoint and let the points* x_0, y_0 *be Nielsen related as fixed points of* f^k *(where* $l|k$*). Moreover, let* S_r^l *and* $S_{r'}^k$ *be satisfied at* y_0 *and* x_0 *respectively. Then there is a homotopy* f_t *constant in a neighbourhood of* $\mathrm{Fix}(f^k) \setminus \{x_0, ..., x_{k-1}; y_0, ..., y_{l-1}\}$ *satisfying* $f_0 = f$, $\mathrm{Fix}(f_1^k) = \mathrm{Fix}(f^k) \setminus \{x_0, ..., x_{k-1}\}$.

3. The least number of periodic points of the given minimal periods

The next Theorem is the extension of Theorem 3.3 from [JKM] to dimension 3. This theorem shows that the algebraic necessary condition for $n \in \mathrm{HPer}(f)$ is also sufficient.

THEOREM (3.1). *Let $f\colon M \to M$ be a self-map of a compact PL-manifold of dimension $d \geq 3$. Then f is homotopic to a map g satisfying $P_n(g) = \emptyset \iff$ there is no essential irreducible Reidemeister class in $\mathcal{R}(f^n)$.*

Proof. $\Rightarrow$ is evident.

It remains to prove $\Leftarrow$. By Theorem (2.2) we may assume that $\mathrm{Fix}(f^n)$ is finite and moreover f is a local PL-homeomorphism near each fixed point of f^n. Let us denote $\mathrm{Fix}(f^n) = A \cup B$ where A denotes the sum of all irreducible classes and B the sum of all reducible ones.

Lemma (3.2) gives a homotopy $\{f_t\}_{0\leq t\leq 1}$ which is constant in a neighbourhood of $\mathrm{Fix}(f^n)\setminus A = B$, $f_0 = f$ and $\mathrm{Fix}(f_1^n) = \mathrm{Fix}(f_0^n)\setminus A$. Since the homotopy is constant in $\mathrm{Fix}(f_1^n)$, all classes in $\mathrm{Fix}(f_1^n)$ remain reducible. Now Lemma (3.4) yields a homotopy from f_1 to a map g satisfying $P_n(g) = \emptyset$.

The proof of Theorem (3.1) will be complete once Lemmas (3.2) and (3.4) are proved. □

LEMMA (3.2). *Let $f\colon M \to M$ be a self-map of a compact PL-manifold with no essential and irreducible classes in $\mathrm{Fix}(f^n)$. Moreover we assume that $\mathrm{Fix}(f^n)$ is finite and f is a local PL-homeomorphism near each fixed point of f^n. Let A be the sum of all irreducible Nielsen classes of f^n, i.e., classes of depth $= n$. Then there is a homotopy $\{f_t\}$ constant in a neighbourhood of $\mathrm{Fix}(f^n) \setminus A$ and satisfying $f_0 = f$, $\mathrm{Fix}(f_1^n) = \mathrm{Fix}(f_0^n) \setminus A$.*

Proof. Since each orbit in A is irreducible, the length of each orbit of points in A is n. Moreover the orbits of classes in A are inessential hence their sum splits into pairs of orbits of length n of opposite indices ± 1. Consider a pair of orbits of points $\{x_1, ..., x_n; y_1, ..., y_n\} \subset A$ of opposite indices. The Cancelling Procedure yields a homotopy constant in a neighbourhood of $\mathrm{Fix}(f^n)\setminus\{x_1, ..., x_n; y_1, ..., y_n\}$ such that $f_0 = f$ and $\mathrm{Fix}(f_1^n) = \mathrm{Fix}(f^n) \setminus \{x_1, ..., x_n; y_1, ..., y_n\}$. Following this procedure we can reduce the number of such orbits to zero, hence we get $\mathrm{Fix}(f_1^n) = \mathrm{Fix}(f_0^n) \setminus A$ as required. □

Remark (3.3). If f_1 satisfies the above Lemma, then all the Nielsen classes in $\mathrm{Fix}(f_1^n)$ are reducible. □

LEMMA (3.4). *If all Nielsen classes in $P_n(f)$ are reducible, then f is homotopic to a map g satisfying $P_n(g) = \emptyset$.*

Proof. By Theorem (2.2) we may assume that $\mathrm{Fix}(f^n)$ is finite and f is a PL-homeomorphism near each fixed point of f^n. If each orbit of points in $\mathrm{Fix}(f^n)$ has length $< n$ then $P_n(f) = \emptyset$. Now we assume that $\mathrm{Fix}(f^n)$ contains exactly one orbit $\{a_0, ..., a_{n-1}\}$ of length n. By assumption the orbit of the Nielsen classes containing $\{a_0, ..., a_{n-1}\}$ is reducible; hence we may assume that there is also an orbit $\{b_0, ..., b_k\}$ of length $k < n$ in this class. In general such a orbit may not exist, but then we may apply the Addition Formula and create such orbit. Now we may apply the Coalescing Procedure and we get a homotopy constant on $\mathrm{Fix}(f^n) \setminus \{a_0, ..., a_{n-1}\}$ such that $f_0 = f$, $\mathrm{Fix}(f_1^n) = \mathrm{Fix}(f^n) \setminus \{a_0, ..., a_{n-1}\}$. Then all orbits of points in $\mathrm{Fix}(f_1^n)$ have length $< n$; hence $P_n(f_1) = \emptyset$.

In general $\mathrm{Fix}(f^n)$ may contain several orbits of length n. Then we use the arguments from the last section of [Je4] to coalesce simultaneously all these orbits to shorter ones. □

One might expect that such an operation of removing $P_k(f)$ can be done simultaneously for two or more periods. But the next example shows that this is not possible in general: we can not remove points of periods 1 and 2 simultaneously although there is no essential irreducible orbit.

Example (3.5). Let $f\colon S^{2n} \to S^{2n}$ be the antipodal map $f(x) = -x$. Since S^{2n} is simply connected, $\mathcal{R}(f^k)$ consists of one class for each $k \in \mathbb{N}$. In particular the unique orbit in $\mathcal{R}(f^2)$ reduces to the only class in $\mathcal{R}(f^1)$. But the last one is inessential since $\mathrm{Fix}(f) = \emptyset$. Thus there is no essential irreducible orbit. Nevertheless $\mathrm{Fix}(g^2) \neq \emptyset$ for each $g \sim f$ since $g^2 \sim f^2 = \mathrm{id}$ and $L(\mathrm{id}) = \chi(S^{2n}) = 2$. □

The next Theorem makes precise for which periods it is possible to remove periodic points simultaneously.

Theorem (3.6). *Let $f\colon M \to M$ be a selfmap of a compact PL-manifold of dimension ≥ 3. Let $N_0 \subset \mathbb{N}$ be finite. Then there is a homotopy $f_t\colon M \to M$ such that $f_0 = f$ and $\mathrm{P}_r(f_1) = \emptyset$ for all $r \in N_0$ if and only if for every $r \in N_0$ any essential Reidemeister class $A^r \in \mathcal{R}(f^r)$ reduces to a class $B^s \in \mathcal{R}(f^s)$ for an $s \notin N_0$.*

Proof. $\Rightarrow$ Assume that $\mathrm{P}_r(f) = \emptyset$ for all $r \in N_0$. Consider an essential Reidemeister class $A^r \in \mathcal{OR}(f^r)$ where $r \in N_0$. Since A^r is essential, it contains a point x_0 for an $s|r$. Then $P_s(f) \neq \emptyset$ hence $s \notin N_0$. Now the Reidemeister class A^r reduces to the class $B^s \in \mathcal{R}(f^s)$ represented by the point x_0.

$\Leftarrow$ We use induction with the respect to the number $l = \#N_0$. For $l = 1$ the Theorem follows from Theorem (3.1). Now we assume that the Theorem holds for $< l$. Let $N_0 \subset \mathbb{N}$ be a subset of cardinality l. Let r be the greatest element in N_0. By inductive assumption f is homotopic to a map f_1 satisfying $\mathrm{P}_s(f_1) = \emptyset$ for all $s \in N_0 \setminus \{r\}$. It remains to remove all orbits from $\mathrm{P_r}(f_1)$. Let $A^r \in \mathcal{R}(f_1^r)$ be nonempty.

Suppose that A^r does not reduce (as a Reidemeister class) to any class B^s with $s \notin N_0$. Then A^r is inessential and each orbit of points in A^r must be of length r. Now A^r splits into pairs of orbits of points of length of opposite indices. We may apply the Cancelling Procedure to remove A^r.

Now we suppose that A^r reduces to an orbit of Reidemeister classes $B^s \in \mathcal{OR}(f_1^s)$. If $B^s \neq \emptyset$ then we may apply the Coalescing Procedure and we match A^r to B^s. If $B^s = \emptyset$ then we apply the Addition Property and we create a new orbit of length s representing B^s and we may apply the Coalescing Procedure. Thus each $A^r \in \mathcal{OR}(f_1^r)$ may be either removed or coalesced to an orbit or points of length $s \notin N_0$. Following the arguments from the end of the proof of the Coalescing Procedure we deduce that these operations can be done simultaneously to all orbits of Reidemeister classes in $\mathcal{OR}(f_1^r)$. After this $P_s(f_1) = \emptyset$ for all $s \in N_0$. □

Let $f\colon X \to X$. We say that a subset $N_0 \subset \mathbb{N}$ is *free of f-homotopy minimal periods* if f is homotopic to a map g with $P_s(g) = \emptyset$ for all $s \in N_0$.

4. Nielsen number for selfmaps of $\mathbb{R}P^n$

We start with a classification of the homotopy classes of selfmaps of a real projective space $\mathbb{R}P^n$ for $n \geq 2$. We notice that $\pi_k(\mathbb{R}P^n) = \pi_k(S^n) = 0$ for $2 \leq k < n$; hence if two maps $f, g\colon \mathbb{R}P^n \to \mathbb{R}P^n$ induce the same homotopy homomorphisms $f_\# = g_\#$ of $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$, then g is homotopic to a map whose restriction to $(n-1)-$skeleton $\mathbb{R}P^{n-1} \subset \mathbb{R}P^n$ equals f. Thus we may assume that $g = f\#c$ where $\#$ denotes the connected sum with a map $c\colon S^n \to \mathbb{R}P^n$. Since $\pi_n(\mathbb{R}P^n) = \pi_n(S^n) = \mathbb{Z}$ and there are two homomorphisms of the group $\mathbb{Z}_2$, any selfmap of the projective space is homotopic to one of the maps f_k, g_k $(k \in \mathbb{Z})$ defined below.

Let $p\colon S^n \to \mathbb{R}P^n$ denote the universal covering. Let $\tilde{x}_0 = (1, 0, ..., 0) \in S^n$, $x_0 = p(\tilde{x}_0) \in \mathbb{R}P^n$ be chosen points. We define $f_k\colon \mathbb{R}P^n \to \mathbb{R}P^n$ as the map induced by an odd map $\tilde{f}_k\colon S^n \to S^n$ satisfying $\tilde{f}_k(\tilde{x}_0) = \tilde{x}_0$ and $\deg(\tilde{f}_k) = 2k+1$. We define g_k as the composition $\mathbb{R}P^n \xrightarrow{r} S^n \xrightarrow{h_k} S^n \xrightarrow{p} \mathbb{R}P^n$ where r sends the $n-1$ dimensional skeleton into $\tilde{x}_0$ and the unique n dimensional cell homeomorphically onto $S^n \setminus \tilde{x}_0$, and h_k is a map of degree k.

Then the set of homotopy classes of self-maps preserving the chosen point of $(\mathbb{R}P^n, x_0)$ may described as follows.

LEMMA (4.1). *If $h_\#\colon \pi_1\mathbb{R}P^n \to \pi_1\mathbb{R}P^n$ is iso then $h = f_k$; and if $h_\#$ is zero then $h = g_k$ for a number $k \in \mathbb{Z}$.* □

Remark (4.2). If n is even then the maps f_k and f_{k+2} are homotopic. If n is odd then $\mathbb{R}P^n$ is orientable; hence deg f is defined. In general the lifts $\tilde{f}$ and $\tilde{f}$ satisfy deg $(-f) = (-1)^{n+1}$deg f; hence the degree of a lift is defined up to sign.

Now we recall the formulae for the Nielsen numbers (Corollaries 5.1 and 6.1 in [JeR]).

LEMMA (4.3). *Let n be even. Then*

$$N(f) = \begin{cases} 2 & \text{for } f_\# = \mathrm{id} \text{ and } f \text{ not homotopic to id,} \\ 1 & \text{otherwise.} \end{cases}$$

Now let n be odd. Then

$$N(f) = \begin{cases} 0 & \text{for } \deg f = 1, \\ 2 & \text{for } \deg f \neq 1 \text{ and } f_\# = \mathrm{id}, \\ 1 & \text{for } \deg f \neq 1 \text{ and } f_\# \neq \mathrm{id}\,. \end{cases}$$

5. Homotopy periods for self-maps of $\mathbb{R}P^n$

Let us note that if a map $f\colon X \to X$ maps the base point into itself, then the induced map between the sets of Reidemeister classes $i_{kl}\colon \mathcal{R}(f^l) \to \mathcal{R}(f^k)$ is given by the formula $i_{k,l}[\alpha] = [\alpha \cdot f^l\alpha \cdot f^{2l}\alpha \cdots f^{k-l}\alpha]$.

We consider a map $f\colon \mathbb{R}P^n \to \mathbb{R}P^n$ $n \geq 2$.

LEMMA (5.1). *If $f_\# = \mathrm{id}$ then $\mathcal{R}(f^k) = \pi_i(\mathbb{R}P^n) = \mathbb{Z}_2$ and the map $i_{k,l}\colon \mathcal{R}(f^l) \to \mathcal{R}(f^k)$ is bijective for k/l odd and is constant if k/l is even.*

Proof. $i_{k,l}[\alpha] = [\alpha \cdot f^l\alpha \cdot f^{2l}\alpha \cdots f^{k-l}\alpha] = [\alpha^{k/l}]$. Since $\pi_1\mathbb{R}P^n = \mathbb{Z}_2$, the lemma follows.

COROLLARY (5.2). *Let* $f\colon \mathbb{R}P^n \to \mathbb{R}P^n$ *satisfy* $N(f^k) = 2$ *for all* $k \in \mathbb{N}$. *Then* $\mathrm{HPer}\,(f) = \{2^k; k = 0, 1, 2, ...\}$.

Proof. $N(f) = 2$ implies $\#\mathcal{R}(f) = 2$ and $f_\# = \mathrm{id}$. If k is not of the form 2^l then $k = rs$ where $s > 1$ is odd. By the above lemma $i_{k,r}\colon \mathcal{R}(f^r) \to \mathcal{R}(f^k)$ is bijective; hence both classes in $\mathcal{R}(f^k)$ reduce to $\mathcal{R}(f^r)$ and $k \notin \mathrm{HPer}\,(f)$.

Now let $k = 2^l$. Since for any $r|k$, $r < k$ the number k/r is even, the image of $i_{k,r}$ is a point. This point is the same for all divisors since $\mathrm{im} i_{k,1} \subset \mathrm{im} i_{k,r}$. Now the other element of $\mathcal{R}(f^k)$ is an irreducible essential class; hence $k \in \mathrm{HPer}\,(f)$. □

LEMMA (5.3). *If* $f^2 = \mathrm{id}$ *and* $N(f) \neq 0$ *then* $\mathrm{HPer}\,(f) = \{1\}$.

Proof. Notice that $N(f) \neq 0$ implies $1 \in \mathrm{HPer}\ f$. On the other hand $f^2 = \mathrm{id}$ implies $f_\# = \mathrm{id}$. If k is odd then $i_{k,1}$ is epi. If k is even then there is at most one essential Nielsen class of f^k, since $\mathrm{Fix}(f^k) = \mathrm{Fix}(\mathrm{id}) = \mathbb{R}P^n$. Now any essential Nielsen class of f precedes the essential class in $\mathcal{R}(f^k)$; hence $k \notin \mathrm{HPer}\,(f)$

Now we are in a position to prove the main result. □

THEOREM (5.4). *Let* $f\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$, $n \geq 3$. *Then*

$N(f) = 0 \implies \mathrm{HPer}\,(f) = \emptyset$,

$N(f) = 1 \implies \mathrm{HPer}\,(f) = \{1\}$,

$N(f) = 2 \implies \mathrm{HPer}\,(f) = \{1, 2, 2^2, 2^3, ...\}$, *with one exception: for n-odd,* $\deg(f) = -1$, $f_\# = \mathrm{id}$,

$N(f) = 2$, *but* $\mathrm{HPer}\,(f) = \{1\}$.

Proof. We consider cases:

$N(f) = 0$. This is satisfied iff n is odd and $\deg\ f = 1$. But then for every $k \in \mathbb{N}$ also $\deg\ f^k = 1$ which implies $N(f^k) = 0$; hence there are no essential classes and $\mathrm{HPer}\ f = \emptyset$.

$N(f) = 1$. Let n be even. Then either $f_\# \neq \mathrm{id}$ or $f = \mathrm{id}$. We recall that $\mathbb{R}P^n$ is $\mathbb{Q}$-acyclic (n is even) hence the Lefschetz number $L(f) \neq 0$ for every selfmap of $\mathbb{R}P^n$. Thus $1 \in \mathrm{HPer}\,(f)$ for every f. It remains to show that no other natural number belongs to $\mathrm{HPer}\,(f)$. This is evident for $f = \mathrm{id}$. Otherwise $f_\# \neq \mathrm{id}$ hence $f_\#^k \neq \mathrm{id}$ and there is only one Reidemeister class for each $k \in \mathbb{N}$, All these classes reduce to 1.

Let n be odd. Then $\deg\ f \neq 1$ and $f_\# \neq \mathrm{id}$. Since $\deg\ f \neq 1$, $L(f) = 1 - \deg(f) \neq 0$ and $1 \in \mathrm{HPer}\ f$. On the other hand $f_\# \neq \mathrm{id}$ implies $f_\#^k \neq \mathrm{id}$; hence there is only one Reidemeister class for each k and no $k > 1$ belongs to $\mathrm{HPer}\ f$.

$N(f) = 2$. Let n be even. Then $f_\# = \mathrm{id}$ and f is not homotopic to the identity. We will show that $N(f^k) = 2$ for all $k \in \mathbb{N}$ and then the lemma will follow from Lemma (5.2). We notice that $f_\# = \mathrm{id}$ means that f is induced by a map $\tilde{f}\colon S^n \to S^n$ of odd degree. Since $f \neq \mathrm{id}$, $\deg\tilde{f} \neq \pm 1$. Thus $f_\#^k = \mathrm{id}$

and $\deg\tilde{f}^k \neq \pm 1$ implies f^k not homotopic to the identity. Thus $N(f^k) = 2$ by Lemma (4.3).

Now let n be odd. Then $f_\# = \mathrm{id}$ and $\deg f \neq 1$.

Let us assume moreover that $\deg f \neq \pm 1$. Then we may follow the above to prove that $N(f^k) = 2$ for all $k \in \mathbb{N}$ and then the lemma follows from Lemma (5.2).

It remains to consider the case $\deg(f) = -1$, $f_\# = \mathrm{id}$. Since then $f^k_\# = \mathrm{id}$, $\#\mathcal{R}(f^k) = 2$ for all $k \in \mathbb{N}$. If f is odd then $i_{k1}\colon \mathcal{R}(f) \to \mathcal{R}(f^k)$ is epi (Lemma (5.1)) hence $k \notin \mathrm{HPer}\ f$. If k is even, then $\deg f^k = 1$; hence $f^k = \mathrm{id}$ and $N(f^k) = N(\mathrm{id}) = 0$ and $k \notin \mathrm{HPer}\ f$. Thus $\mathrm{HPer}\ f = \{1\}$. The last is the exceptional case. □

The next Corollary shows that the situation as in Example (3.5) does not occur in the case of projective spaces.

COROLLARY (5.5). *Let* $f\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$, $n \geq 3$. *Then each finite subset* $N_0 \subset \mathbb{N}$ *disjoint from* $\mathrm{HPer}\,(f)$ *is free of f-homotopy minimal periods.*

Proof. We will refer to the proof of Theorem (5.4).

Let $\mathrm{HPer}\,(f) = \emptyset$. Then $N(f^k) = 0$ for all $k \in \mathbb{N}$; hence the assumptions of Theorem (5.4) are satisfied for each $N_0 \subset \mathbb{N}$.

Let $\mathrm{HPer}\,(f) = \{1\}$. If moreover $N(f) = 1$ then either $f_\# = 0$ or $f \sim \mathrm{id}$. It remains to show that in the both cases all essential classes reduce to a class in $\mathcal{R}(f^1)$. If $f_\# = 0$ then $\mathcal{R}(f^k)$ consists of a single element for each $k \in \mathbb{N}$. All these elements are preceded by the single element in $\mathcal{R}(f^1)$. If $f = \mathrm{id}$ then $\mathcal{R}(f^k)$ so there is exactly one nonempty Nielsen class in $\mathcal{R}(f^k)$ for each $k \in \mathbb{N}$, and these classes reduce to the unique nonempty class in $\mathcal{R}(f^1)$.

Now we consider the exceptional case: n is odd, $\deg(f) = -1$, $f_\# = \mathrm{id}$. Then, as we have noticed there is no essential class in $\mathcal{R}(f^k)$ for k even. On the other hand $i_{k1}\colon \mathcal{R}(f^1) \to \mathcal{R}(f^k)$ is epi for k odd. Thus each essential Nielsen class reduces to $\mathcal{R}(f^1)$.

Let $\mathrm{HPer}\,(f) = \{1, 2, 2^2, 2^3, \dots\}$. We notice that the assumptions of Theorem (3.6) are satisfied, for each N_0, if each essential class $A \in \mathcal{R}(f^k)$, $k \notin \mathrm{HPer}\,(f)$, reduces to a class in $\mathcal{R}(f^l)$, $l \in \mathrm{HPer}\,(f)$. In our case if $k \notin \mathrm{HPer}\,(f)$ then $k = r \cdot 2^s$ where r is odd. But then $i_{k,2^s}\colon \mathcal{R}(f^{2^s}) \to \mathcal{R}(f^k)$ is into, hence all the classes in $\mathcal{R}(f^k)$ reduce to $\mathcal{R}(f^{2^s})$. □

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INSTITUTE OF APPLIED MATHEMATICS
UNIVERSITY OF AGRICULTURE
NOWOURSYNOWSKA 159
02 757 WARSZAWA
POLAND
jezierski@alpha.sggw.waw.pl

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INDICE DE MASLOV ET THÉORÈME DE NOVIKOV-WALL

PIERRE PY

Abstract. This paper deals with the link between the signature of 4-manifolds and the ternary Maslov index in symplectic geometry. We give a proof of a theorem by C.T.C. Wall, which generalizes Novikov's classical additivity theorem. This theorem was formulated and proved by Wall before the appearance of the Maslov index in symplectic geometry. We can now formulate it in a slightly different way. We show how this theorem allows us to compute the signature of a closed oriented manifold using a Morse function. We also give a proof of a theorem by W. Meyer about the signature of fibre bundles over surfaces. Finally, we include a geometric definition of Maslov's index, due to Arnold.

1. Introduction

Le but de ce texte est de décrire quelques interactions entre deux notions d'origines différentes : *l'indice de Maslov* d'une part, issu de la géométrie symplectique, et la *signature* des variétés de dimension 4 d'autre part, de nature topologique. Ce lien a été observé initialement en 1969 par C.T.C. Wall dans [26]. Nous décrivons notamment un théorème de Wall qui généralise l'additivité de la signature de Novikov, ainsi qu'un théorème de W. Meyer sur la signature des variétés de dimension 4 fibrées en surfaces au-dessus d'une surface.

La signature $\sigma(M)$ d'une variété, compacte, orientée, de dimension 4, est la signature de la forme quadratique $H_2(M,\mathbb{R}) \times H_2(M,\mathbb{R}) \to \mathbb{R}$ donnée par l'intersection. Ce lien a été remarqué en premier par Wall alors que l'indice de Maslov n'était pas encore apparu dans le cadre de la géométrie symplectique, et qu'il ne portait pas encore ce nom. Il a montré comment cette notion algébrique intervenait dans le calcul de la signature des variétés. Rappelons d'abord le théorème de S. Novikov sur l'additivité de la signature. Supposons que M soit une variété, compacte, orientée, de dimension 4, obtenue en recollant, par un homéomorphisme qui inverse l'orientation du bord, deux variétés M_1 et M_2 le long d'une réunion de composantes connexes de leurs bords. Alors la signature de M est la somme des signatures de M_1 et M_2. Wall a prouvé dans [26] une généralisation de ce résultat dans le cas où l'on identifie M_1 et M_2 le long d'une *sous-variété à bord* X_0 de leur bord. La sous-variété X_0 possède elle-même un bord Σ de dimension 2. On suppose que les orientations de M_1 et M_2 sont induites par celle de M. L'orientation de M_1 induit une orientation de son bord, donc de X_0, et l'orientation de X_0 induit une orientation de Σ. Dans M, la surface Σ peut être vue comme le bord de trois variétés de dimension 3 distinctes : X_0,

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$X_1 = \partial M_1 \setminus X_0$ et $X_2 = \partial M_2 \setminus X_0$. Les noyaux L_i des applications

$$H_1(\Sigma, \mathbb{R}) \to H_1(X_i, \mathbb{R})$$

fournissent trois sous-espaces de $H_1(\Sigma, \mathbb{R})$, lagrangiens pour la forme d'intersection. La signature n'est plus additive mais le défaut $\sigma(M) - \sigma(M_1) - \sigma(M_2)$ s'interprète comme l'indice de Maslov des trois lagrangiens L_0, L_1, L_2. On a précisément :

Théorème (1.1) (Wall). *Les variétés M, M_1, M_2 et Σ étant orientées comme indiqué ci-dessus, on a :*

$$\sigma(M) = \sigma(M_1) + \sigma(M_2) + \tau(L_1, L_0, L_2)$$

où $\tau(L_1, L_0, L_2)$ désigne l'indice de Maslov des lagrangiens L_1, L_0, L_2 dans l'espace vectoriel $H_1(\Sigma, \mathbb{R})$ muni d'une structure symplectique grâce à la forme d'intersection.

Nous appellerons désormais *théorème de Novikov-Wall* ce résultat. Bien sûr ce théorème reste vrai en dimension $4n$, et nous travaillerons, comme Wall, dans ce cadre. La formulation du théorème de Wall en termes d'indice de Maslov apparaît déjà dans le travail de S.E. Cappell, R. Lee et E.Y. Miller [4]. Cette version raffinée de l'additivité de Novikov a été utilisée par J.-M. Gambaudo et É. Ghys pour calculer le cobord de la fonction signature sur le groupe de tresses B_n [10]. On pourra consulter [11] pour plus de détails sur le lien entre la signature des tresses et celle des variétés. Le théorème de Novikov-Wall permet également de donner une nouvelle preuve d'un théorème de W. Meyer. Dans [17], Meyer a calculé la signature d'une variété M de dimension 4 fibrée en surfaces (fermées) au-dessus d'un pantalon P. Les deux générateurs a et b du groupe fondamental de P agissent sur l'homologie de dimension 1 de la fibre S. Meyer a prouvé que la signature de M ne dépendait que de cette action. Celle-ci préserve la forme d'intersection sur $H_1(S, \mathbb{R})$. En choisissant une base symplectique de $H_1(S, \mathbb{R})$, les lacets a et b fournissent donc deux éléments γ_a et γ_b du groupe symplectique $\mathrm{Sp}(2n, \mathbb{R})$, définis à conjugaison près. On introduira alors au paragraphe 3.3, un 2-cocycle borné sur le groupe symplectique

$$\mathrm{Meyer} : \mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{Sp}(2n, \mathbb{R}) \to \mathbb{Z}$$

et on prouvera, grâce au théorème de Novikov-Wall :

Théorème (1.2) (Meyer). *La signature de M ne dépend que de l'action des générateurs du groupe fondamental de P sur l'homologie de dimension 1 de la fibre. On a précisément :*

$$\sigma(M) = \mathrm{Meyer}(\gamma_a^{-1}, \gamma_b).$$

Cette application du théorème de Novikov-Wall figurait déjà dans [10]. On démontrera en fait le théorème de Meyer dans le cas où M est une variété de dimension $4n$ fibrée au-dessus d'un pantalon. Nous utilisons également le théorème de Novikov-Wall pour prouver le résultat suivant.

Théorème (1.3). *Supposons que M soit une variété différentiable compacte sans bord, de dimension $4n$ et orientée. Soit $f : M \to \mathbb{R}$ une fonction de Morse de points critiques $x_1, \cdots, x_p$. On suppose les valeurs critiques $\lambda_i = f(x_i)$ toutes* *distinctes. Alors, à chaque point critique x_i d'indice $2n$, on peut associer un entier $s(x_i)$ de $\{-1, 0, 1\}$ tel que la signature de M vérifie :*

$$\sigma(M) = \sum s(x_i)$$

où la somme porte sur les points critiques d'indice $2n$. L'entier $s(x_i)$ est la signature de la variété $f^{-1}([\lambda_i - \epsilon, \lambda_i + \epsilon])$ pour tout ϵ suffisamment petit pour que l'intervalle $[\lambda_i - \epsilon, \lambda_i + \epsilon]$ ne contienne que λ_i comme valeur critique. On décrit de plus quand $s(x_i)$ prend chacune des valeurs -1, 0, $+1$.

L'intérêt du théorème de Novikov-Wall est qu'il permet un calcul plus effectif de la signature. Avec la version classique de l'additivité de Novikov, on ne peut espérer calculer la signature d'une variété en la décomposant en morceaux élémentaires de signatures nulles. On n'obtiendrait ainsi que des variétés de signature nulle. Avec le théorème de Novikov-Wall ceci est possible, car on exprime la signature d'une variété en fonction de la signature de ses différents morceaux, et d'un terme supplémentaire. Par exemple, dans le théorème précédent, le calcul de la signature de M à partir de f se fait en décomposant entièrement la variété en sous-variétés à bord de signatures nulles. Les termes non-nuls s'expriment comme des indices de Maslov.

Une autre manière de généraliser le théorème de Novikov est la suivante. On étudie toujours une variété compacte orientée de dimension 4. Considérons le groupe d'homologie $H_2(M, \mathbb{K})$, où $\mathbb{K}$ est un corps quelconque (disons de caractéristique distincte de 2). La forme d'intersection $H_2(M, \mathbb{K}) \times H_2(M, \mathbb{K}) \to \mathbb{K}$ définit un élément $\sigma(M)$ du groupe de Witt de $\mathbb{K}$, noté $W(\mathbb{K})$. Dans le cas où le corps est celui des réels, le groupe de Witt s'identifie à $\mathbb{Z}$ via la signature, et $\sigma(M)$ s'identifie à la signature usuelle de la variété sous cet isomorphisme. On peut encore énoncer et démontrer l'additivité de Novikov, ainsi que sa généralisation par Wall, dans ce cadre. L'additivité a lieu cette fois-ci dans le groupe $W(\mathbb{K})$. Ce point de vue avait déjà été adopté par F. Latour dans [12].

L'organisation du texte est la suivante. Dans la partie 2, après un rappel sur le groupe de Witt, on définit l'indice de Maslov de trois lagrangiens dans un espace vectoriel symplectique sur un corps $\mathbb{K}$. On prouve qu'il vérifie une relation de cocycle et on décrit différentes manières de le représenter par une forme quadratique. Dans la troisième partie on prouve le théorème de Novikov-Wall en se plaçant dans le cas d'un corps quelconque (de caractéristique distincte de 2). On montre ensuite comment calculer la signature d'une variété de dimension 4 fibrée au-dessus d'un pantalon. On prouve enfin le théorème relatif au calcul de la signature à partir d'une fonction de Morse. Dans la dernière partie nous incluons une définition géométrique de l'indice de Maslov (dans le cas où $\mathbb{K} = \mathbb{R}$) due à Arnold. On pourra également consulter les articles de P. Dazord et M. De Gosson [8], [9], où cette approche est discutée en détails. Elle donne un autre point de vue sur l'indice de Maslov que celui de la seconde partie. C'est d'ailleurs ce point de vue qui apparaît lors de l'utilisation de l'indice de Maslov en topologie symplectique [22]. Cette partie est donc essentiellement indépendante des précédentes et sa lecture pourra être omise par un lecteur uniquement intéressé par la question de la signature des variétés. Pour obtenir cette définition géométrique de l'indice de Maslov, on étudie la topologie de la grassmannienne lagrangienne de $\mathbb{R}^{2n}$. Ceci permet de construire un 2-cocycle borné à valeurs réelles qui représente l'image dans $H^2(\mathrm{Sp}(2n,\mathbb{R})^\delta,\mathbb{R})$ de la classe de l'extension centrale

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \longrightarrow \mathrm{Sp}(2n,\mathbb{R}) \longrightarrow 0$$

du groupe symplectique par son revêtement universel.

A l'exception du Théorème (1.3) ci-dessus, aucun des résultats évoqués dans ce texte n'est nouveau. Nous avons indiqué tout au long du texte l'origine de chacun des résultats mentionnés.

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2. Groupe de Witt et indice de Maslov de trois lagrangiens

(2.1) Groupe de Witt. On rappelle ici brièvement la construction du groupe de Witt d'un corps $\mathbb{K}$. Nous supposerons toujours que $\mathbb{K}$ est de caractéristique distincte de 2. On pourra consulter [20] pour plus de détails, et pour la construction dans le cas d'un anneau.

On considère l'ensemble $\mathcal{W}$ des classes d'isomorphisme de $\mathbb{K}$-espaces vectoriels de dimension finie munis d'une forme bilinéaire symétrique non-dégénérée. Un élément V de $\mathcal{W}$ sera dit *hyperbolique* s'il possède un sous-espace $S \subset V$ tel que $S^\perp = S$. On peut voir facilement qu'un espace est hyperbolique si et seulement si il existe une base dans laquelle sa matrice est de la forme

$$\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

On a sur $\mathcal{W}$ une addition naturelle qui, à deux espaces V_1 et V_2, associe la somme directe orthogonale $V_1 \oplus V_2$. En outre, on peut définir sur $\mathcal{W}$ la relation d'équivalence suivante : $V_1 \sim V_2$ s'il existe deux espaces hyperboliques S_1 et S_2 tels que $V_1 \oplus S_1$ et $V_2 \oplus S_2$ soient isomorphes. On note $W(\mathbb{K})$ le quotient de $\mathcal{W}$ par cette relation. L'addition de $\mathcal{W}$ passe au quotient et fait de $W(\mathbb{K})$ un groupe abélien. C'est *le groupe de Witt* de $\mathbb{K}$. En utilisant le produit tensoriel, on peut également le munir d'une structure d'anneau.

Dans le cas du corps des nombres réels, on a un isomorphisme naturel entre $W(\mathbb{R})$ et $\mathbb{Z}$, via la signature. Dans ce qui suit la signature d'une forme bilinéaire symétrique f sur un espace vectoriel réel désignera toujours l'entier $\nu_+ - \nu_-$, où ν_+ est le nombre de valeurs propres positives de f et ν_- le nombre de valeurs propres négatives. Il n'est pas difficile de voir que les modules hyperboliques sont exactement ceux de signature nulle. Ainsi la signature est bien définie sur $W(\mathbb{R})$ et fournit un isomorphisme entre $W(\mathbb{R})$ et $\mathbb{Z}$.

Dans la suite, si (E,f) est un espace vectoriel sur $\mathbb{K}$ muni d'une forme bilinéaire symétrique éventuellement dégénérée, on parlera de l'image de (E,f) dans $W(\mathbb{K})$, notée $\sigma(E)$, pour désigner la classe de E/R où R est le radical de f.

(2.2) Cocycle de Maslov. Il existe de nombreux textes qui introduisent l'indice de Maslov ([2], [8], [9], [15], [21], [23] par exemple). A titre de remarque, si l'on utilise ici l'indice de Maslov pour l'étude de la signature des variétés, cette notion intervient également dans d'autres cadres : pour la résolution d'équations différentielles ou aux dérivées partielles [1], [2], [13], [14], ou encore en topologie symplectique [22].

Signalons également que l'indice de Maslov (dans le cas où $\mathbb{K} = \mathbb{R}$) désigne en fait, selon les situations et les auteurs, un nombre attaché à :

– un triplet de lagrangiens,

– une paire de points dans le revêtement universel de la grassmannienne lagrangienne,

– une paire de chemins dans le groupe symplectique.

Dans ce texte, l'indice de Maslov désignera un entier attaché à un triplet de lagrangiens. Cet entier est parfois également appelé *signature de Kashiwara*. Le troisième cas ci-dessus n'apparaîtra pas ici, le second sera décrit dans la partie 4. Dans [4], on pourra trouver une description complète de chacune des versions de l'indice de Maslov (celles évoquées ci-dessus, mais également d'autres). Signalons également des travaux plus récents [5], [6], [7], dans lesquels l'indice de Maslov est étendu à un cadre plus général.

On suppose donc que (E, ω) est un $\mathbb{K}$-espace vectoriel muni d'une forme symplectique : c'est-à-dire, ω est une forme bilinéaire anti-symétrique non-dégénérée sur E. L'orthogonal d'un sous-espace V de E est :

$$V^{\perp} = \{u \in E,\ \omega(u, v) = 0,\ \forall v \in V\}.$$

Un *lagrangien* de E est un sous-espace L de E isotrope pour ω et de dimension maximale pour cette propriété. Il est équivalent de dire que L est égal à son orthogonal. Enfin on désignera par $\mathrm{Sp}(E, \omega)$ le groupe des automorphismes de E qui préservent la forme ω : ce sont les applications linéaires $g : E \to E$ telles que $\omega(gu, gv) = \omega(u, v)$ pour u et v dans E.

Si L_1,L_2,L_3 sont trois lagrangiens de E, on va définir leur indice de Maslov $\tau(L_1, L_2, L_3)$ comme étant un élément du groupe de Witt de $\mathbb{K}$. On considère dans $L_1 \oplus L_2 \oplus L_3$ le sous espace V formé des vecteurs $v = (v_1, v_2, v_3)$ tels que $v_1 + v_2 + v_3 = 0$. On le munit de la forme bilinéaire symétrique définie par :

$$f(v, v') = \omega(v_2, v'_1) = \omega(v_3, v'_2) = \omega(v_1, v'_3).$$

L'indice de Maslov des trois lagrangiens L_1,L_2,L_3 est alors la classe de V muni de f dans le groupe $W(\mathbb{K})$. Il est bien clair que l'indice de Maslov est invariant sous l'action du groupe symplectique :

$$\tau(L_1, L_2, L_3) = \tau(gL_1, gL_2, gL_3)\,,$$

pour un élément g de $\mathrm{Sp}(E, \omega)$. Il change de signe si l'on échange deux des trois lagrangiens et est invariant par permutations circulaires. En outre l'indice de Maslov vérifie la relation de cocycle suivante.

Théorème (2.2.1). *Si L_1,L_2,L_3,L_4 sont quatre lagrangiens de E on a :*

$$\tau(L_1, L_2, L_3) - \tau(L_1, L_2, L_4) + \tau(L_1, L_3, L_4) - \tau(L_2, L_3, L_4) = 0.$$

Preuve : la preuve suivante est tirée de [25]. On considère le sous-espace U de $L_1 \oplus L_2 \oplus L_3 \oplus L_4$ formé des vecteurs v tels que $v_1 + v_2 + v_3 + v_4 = 0$, que l'on munit de la forme bilinéaire suivante :

$$\begin{aligned}\phi(v, v') = \tfrac{1}{4}(&\omega(v_2, v'_1) + \omega(v_3, v'_2) + \omega(v_4, v'_3) + \omega(v_1, v'_4)\\ &+\omega(v'_2, v_1) + \omega(v'_3, v_2) + \omega(v'_4, v_3) + \omega(v'_1, v_4)).\end{aligned}$$

On va calculer de deux manières différentes la classe de U dans $W(\mathbb{K})$ pour obtenir la relation voulue. On note E_i le sous-espace de U formé des vecteurs tels que $v_i = 0$ $(1 \leq i \leq 4)$. L'espace E_4 s'identifie naturellement à l'espace V considéré précédemment pour définir l'indice de Maslov de L_1,L_2,L_3. De plus si v et v' sont dans E_4 on a :

$$\phi(v, v') = f(v, v').$$

Donc $\sigma(E_4) = \tau(L_1, L_2, L_3)$. On a de la même manière les égalités suivantes :

$$\sigma(E_3) = \tau(L_1, L_2, L_4),\ \sigma(E_2) = \tau(L_1, L_3, L_4),\ \text{et}\ \sigma(E_1) = \tau(L_2, L_3, L_4).$$

La relation de cocycle cherchée se réécrit donc :

$$\sigma(E_2) + \sigma(E_4) = \sigma(E_1) + \sigma(E_3).$$

Pour la prouver on montre que chacun des deux membres est égal à la classe de U dans $W(\mathbb{K})$. Supposons prouvé que l'orthogonal dans U de $E_2 \cap E_4$ est $E_2 + E_4$. Alors par le lemme ci-dessous, $\sigma(U) = \sigma(E_2 + E_4) = \sigma(E_2) + \sigma(E_4)$, la dernière égalité provenant du fait que $E_2 \cap E_4$ est orthogonal à E_2 et E_4. Il nous reste à voir que

$$(E_2 \cap E_4)^{\perp} = E_2 + E_4$$

(l'égalité $\sigma(U) = \sigma(E_1) + \sigma(E_3)$ se prouvant de même). Si a est un vecteur de $L_1 \cap L_3$ on a $\phi(v, (a, 0, -a, 0)) = \frac{1}{2}(\omega(v_2, a) - \omega(v_4, a)) = \omega(v_2, a) = \omega(a, v_4)$. Donc si v est dans E_2 ou E_4 cette dernière quantité est nulle. Si à l'inverse cette quantité est nulle pour tout a de $L_1 \cap L_3$, le vecteur v_2 se trouve dans l'orthogonal de $L_1 \cap L_3$ qui est $L_1 + L_3$ puisque ces deux espaces sont lagrangiens. On note $v_2 = v_{12} + v_{32}$. On peut alors écrire $v = (-v_{12}, v_2, -v_{32}, 0) + (v_1 + v_{12}, 0, v_3 + v_{32}, v_4)$. Le vecteur v est dans $E_2 + E_4$. □

Lemme (2.2.2). *Soit V un espace vectoriel muni d'une forme bilinéaire symétrique. Si S est un sous-espace isotrope de V et L un supplémentaire de S dans $S^{\perp}$ on a dans $W(\mathbb{K})$:*

$$\sigma(V) = \sigma(S^{\perp}) = \sigma(L).$$

Preuve : on le prouve dans le cas où la forme est non-dégénérée. Le cas général s'y ramène en factorisant par le radical de la forme considérée. L'égalité $\sigma(S^{\perp}) = \sigma(L)$ est claire, il nous suffit donc de prouver $\sigma(V) = \sigma(L)$.

Supposons que v soit dans $L \cap L^{\perp}$. En particulier, puisque v est dans L il est orthogonal à S. Donc v est orthogonal à $S \oplus L$, c'est-à-dire dans $S^{\perp\perp} = S$. Le vecteur v est alors nul car $S \cap L = \{0\}$. Donc L et $L^{\perp}$ sont supplémentaires. Pour prouver l'égalité voulue il suffit de voir que $L^{\perp}$ est hyperbolique. Or S est un sous-espace isotrope de $L^{\perp}$. Il suffit de vérifier qu'il est égal à son orthogonal dans $L^{\perp}$ ce qui assure que $L^{\perp}$ est hyperbolique. □

Rappelons maintenant la construction suivante, appelée parfois *contraction* : si F est un sous-espace isotrope de E, le quotient $F^{\perp}/F$ est naturellement muni d'une forme symplectique. En outre si L est un lagrangien de E, il n'est pas difficile de vérifier que $L^F = (L \cap F^{\perp} + F)/F$ est un lagrangien de $F^{\perp}/F$.

Supposons alors que L_1,L_2,L_3 soient trois lagrangiens de E et considérons le sous-espace isotrope suivant :

$$F = L_1 \cap L_2 + L_2 \cap L_3 + L_3 \cap L_1.$$

On note toujours V le sous-espace de $L_1 \oplus L_2 \oplus L_3$ formé des vecteurs dont la somme des trois coordonnées est nulle, et V^F le sous-espace défini de manière analogue dans $L_1^F \oplus L_2^F \oplus L_3^F$. Si v est dans V, chacun des vecteurs v_1,v_2,v_3 est dans $F^\perp$. On dispose donc d'une application naturelle $V \to V^F$. Elle envoie la forme quadratique f sur la forme quadratique correspondante f^F. En outre son noyau est contenu dans le radical de f, donc :

$$\tau(L_1, L_2, L_3) = \tau(L_1^F, L_2^F, L_3^F)\,.$$

Autrement dit, l'indice de Maslov est invariant par cette contraction. Ce résultat peut par exemple être utile pour étendre à tous les triplets de lagrangiens un résultat déjà connu pour les triplets de lagrangiens deux-à-deux transverses. En effet, on peut vérifier que les trois lagrangiens L_1^F,L_2^F,L_3^F sont deux-à-deux transverses.

(2.3) Définitions supplémentaires de l'indice de Maslov. On donne ici d'autres manières de représenter l'indice de Maslov par une certaine forme quadratique. Notamment on fait le lien entre la définition donnée au paragraphe précédent et celle de [15].

Dans [15], l'indice de Maslov est défini comme la classe de la forme quadratique Q définie sur $L_1 \oplus L_2 \oplus L_3$ par :

$$Q(x) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1).$$

Notons provisoirement $\mu(L_1, L_2, L_3)$ l'indice ainsi défini.

Proposition (2.3.1). *Si $E = L_1 \oplus L_3$, alors l'indice μ se représente par la forme quadratique q sur L_2 définie par :*

$$q(x) = \omega(p_1 x, p_3 x).$$

Ici, p_1 (resp. p_3) est la projection sur L_1 (resp. L_3) parallèlement à L_3 (resp. L_1).

Preuve : on considère la transformation de $L_1 \oplus L_2 \oplus L_3$ dans lui-même :

$$y_1 = x_1 - p_1 x_2,\ y_2 = x_2,\ y_3 = x_3 - p_3 x_2.$$

La forme quadratique Q est alors transformée en :

$$\tilde{Q}((y_1, y_2, y_3)) = -\omega(y_1, y_3) + \omega(p_1 y_2, p_3 y_2)\,.$$

Il n'est pas difficile de vérifier que $L_1 \oplus L_3$ muni de la forme $-\omega(y_1, y_3)$ est hyperbolique. On en déduit la proposition. □

Supposons maintenant L_1,L_2,L_3 deux à deux transverses et montrons que les indices τ et μ coïncident. En notant toujours V le sous-espace de $L_1 \oplus L_2 \oplus L_3$ formé des vecteurs dont la somme des coordonnées est nulle dans E, on a un isomorphisme :

$$L_2 \to V$$
$$v \mapsto x_v = (-p_1(v), v, -p_3(v)).$$

On a $\omega(p_1(v), p_3(v)) = \omega(v, p_3(v)) = \omega(v, -p_1(v)) = f(x_v, x_v)$. Donc par la proposition ci-dessus, $\tau(L_1, L_2, L_3) = \mu(L_1, L_2, L_3)$. Ainsi l'indice μ coïncide avec l'indice que nous avons défini. Puisque notre indice est invariant par contraction, il suffit de vérifier que μ l'est également, ce qui assurera que $\mu = \tau$ dans tous les cas. Ceci est impliqué par les résultats de [15] et [21]. Expliquons brièvement pourquoi.

Dans [15] l'invariance par contraction est prouvée dans le cas où $\mathbb{K} = \mathbb{R}$. En fait, en relisant la preuve on constate qu'elle est vraie dès que μ satisfait la relation de cocycle. Celle-ci est satisfaite dès que l'on a la propriété suivante (toujours par [15]) :

Propriété (2.3.2). Si (E,ω) est un espace vectoriel symplectique sur $\mathbb{K}$, et si L_1,L_2,L_3,L_4 sont quatre lagrangiens de E, on peut trouver un cinquième lagrangien transverse à L_1,L_2,L_3 et L_4.

D'après [21] ceci est vrai dès que le corps a au moins quatre éléments. Dans le cas de $\mathbb{F}_3$, la relation de cocycle est encore vraie, en utilisant une extension adéquate et un résultat sur le groupe de Witt (on se réfère à [21] là encore). Donc μ est invariant par contraction quel que soit le corps $\mathbb{K}$ (de caractéristique distincte de 2) et μ coïncide bien avec l'indice de Maslov que nous avons défini. On a donc prouvé la

Proposition (2.3.3). *Pour tout triplet de lagrangiens* (L_1, L_2, L_3)*, on a l'égalité*

$$\tau(L_1, L_2, L_3) = \mu(L_1, L_2, L_3).$$

Enfin, une dernière manière de représenter l'indice de Maslov des trois lagrangiens L_1,L_2,L_3 est de considérer la forme quadratique non-dégénérée sur

$$\frac{L_1 \cap (L_2 + L_3)}{L_1 \cap L_2 + L_1 \cap L_3}$$

qui à un vecteur x représenté par $x_1 = x_2 + x_3$, associe $q(x) = \omega(x_1, x_2)$. Il suit facilement de notre première définition que cette forme quadratique représente l'indice de Maslov $\tau(L_1, L_2, L_3)$.

Le groupe symplectique $\mathrm{Sp}(E,\omega)$ agit transitivement sur l'ensemble des lagrangiens d'une part, et sur l'ensemble des paires de lagrangiens transverses d'autre part. Ce n'est plus le cas pour les triplets de lagrangiens transverses, puisque l'indice de Maslov est invariant par l'action du groupe symplectique. Par contre la définition de la Proposition (2.3.1) permet de montrer que l'indice de Maslov est un invariant complet pour l'action de $\mathrm{Sp}(E,\omega)$ sur les triplets de lagrangiens deux-à-deux transverses.

Proposition (2.3.4). *Soient* L_1,L_2,L_3 *et* L'_1,L'_2,L'_3 *deux triplets de lagrangiens deux à deux transverses. Supposons que* $\tau(L_1, L_2, L_3) = \tau(L'_1, L'_2, L'_3)$*. Alors il existe* g *dans* $\mathrm{Sp}(E,\omega)$ *tel que* $L'_1 = g(L_1)$,$L'_2 = g(L_2)$ *et* $L'_3 = g(L_3)$*.*

Preuve : on utilise donc la Proposition (2.3.1) pour représenter les indices de Maslov par les formes non-dégénérées $q : x \mapsto \omega(p_1(x), p_3(x))$ sur L_2 et $q' : x \mapsto \omega(p'_1(x), p'_3(x))$ sur L'_2. Dire que les deux triplets ont le même indice de Maslov c'est dire qu'il existe deux modules hyperboliques S et S' tels que $L_2 \oplus S \simeq L'_2 \oplus S'$. Alors, S et S' ont nécessairement le même rang, et sont par conséquent isomorphes (tout module hyperbolique de rang $2k$ est isomorphe à la somme directe orthogonale de k plans $\mathbb{K} \oplus \mathbb{K}$ munis de la forme $(x, y) \mapsto 2xy$). Ceci implique alors que L_2 et L'_2, munis de leurs formes quadratiques, sont isomorphes (c'est un cas particulier d'un théorème de Witt, cf. [20] page 8). Fixons un isomorphisme

$$f : L_2 \to L_2'.$$

Soit $v_1, \ldots, v_n$ une base orthogonale de L_2 pour q. Alors $b = (p_1(v_l), p_3(v_l))$ est une base symplectique de E (dans le sens où $\omega(p_1(v_i), p_1(v_j)) = 0$, $\omega(p_3(v_i), p_3(v_j)) = 0$, $\omega(p_1(v_i), p_3(v_j)) = \delta_{ij}\alpha_i$ pour des éléments α_i de $\mathbb{K}^\times$). De même, $b' = (p_1' f(v_l), p_3' f(v_l))$ est une autre base symplectique de E (avec des coefficients α_i identiques). La transformation g qui envoie b sur b' est symplectique et envoie le triplet (L_1, L_2, L_3) sur le triplet (L_1', L_2', L_3'). □

3. Théorème de Novikov-Wall et applications

Tous les groupes de cohomologie (ou d'homologie) que l'on considérera sont des groupes de cohomologie (d'homologie) singulière, à coefficients dans un corps $\mathbb{K}$ fixé.

(3.1) Trois lagrangiens dans $H^{2n-1}(\Sigma)$. On suppose ici que X est une variété compacte, connexe, orientée, de dimension $4n-1$ et de bord Σ. On notera $[X] \in H_{4n-1}(X, \Sigma)$ la classe fondamentale de X et $[\Sigma] = \partial_*[X] \in H_{4n-2}(\Sigma)$ celle de Σ, induite par l'orientation de X. Enfin on désignera par $x \cdot_X y$ et $x \cdot_\Sigma y$ les formes bilinéaires d'intersection

$$H^{2n-1}(X) \times H^{2n}(X, \Sigma) \to \mathbb{K}$$
$$\text{et } \; H^{2n-1}(\Sigma) \times H^{2n-1}(\Sigma) \to \mathbb{K}\,.$$

Par dualité de Poincaré, ces deux formes sont non-dégénérées. La forme $x \cdot_\Sigma y$ est anti-symétrique et munit l'espace $H^{2n-1}(\Sigma)$ d'une structure symplectique.

On considère la partie suivante de la suite exacte longue de cohomologie associée à la paire (X, Σ) :

$$H^{2n-1}(X) \xrightarrow{\pi} H^{2n-1}(\Sigma) \xrightarrow{j} H^{2n}(X, \Sigma).$$

On a le :

Lemme (3.1.1). *Si $a \in H^{2n-1}(\Sigma)$ et $b \in H^{2n-1}(X)$, on a :*

$$a \cdot_\Sigma \pi(b) = b \cdot_X j(a)$$

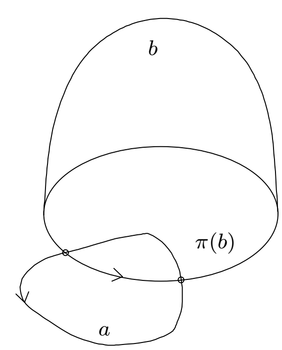

Fig. 1

Preuve : c'est une conséquence de la formule :

$$\partial(x \cup y) = \partial x \cup y + (-1)^{degx} x \cup \partial y.$$

On prend pour x un cocycle sur X qui représente b et pour y une cochaine sur X dont la restriction à Σ est un cocycle représentant a. □

Donnons une brève interprétation géométrique de ce lemme. Supposons que $n = 1$. La variété ambiante X est donc de dimension 3. En utilisant les isomorphismes de dualité de Poincaré on peut donner une version homologique du résultat. L'élément b devient une chaîne de dimension 2 dont le bord $\pi(b)$ est dans la surface Σ. L'élément a est un cycle de dimension 1 dans Σ. Le lemme affirme que le nombre d'intersection de a et de $\pi(b)$ dans la surface Σ, est le même que celui de a et de b dans X (fig. 1).

Une conséquence de ce lemme est que le sous-espace $L = \operatorname{Ker} j = \operatorname{Im} \pi$ est isotrope pour la forme symplectique de $H^{2n-1}(\Sigma)$. En outre, si a est orthogonal à L, on a pour tout b de $H^{2n-1}(X)$, $b \cdot_X j(a) = 0$. Donc $j(a)$ est nul, et a est lui-même dans L. Le sous-espace L est égal à son propre orthogonal, c'est-à-dire est lagrangien. On a donc prouvé le résultat suivant :

Proposition (3.1.2). *L'image de l'application $H^{2n-1}(X) \to H^{2n-1}(\Sigma)$ est un lagrangien de $H^{2n-1}(\Sigma)$.*

Donnons maintenant un argument plus géométrique en faveur de la proposition précédente. En utilisant, là encore, les isomorphismes de dualité de Poincaré, cette proposition affirme que le noyau de l'application $H_{2n-1}(\Sigma) \to H_{2n-1}(X)$ est un lagrangien pour la forme d'intersection de Σ. Le calcul de la dimension de ce noyau peut se faire en utilisant des arguments de dualité de Poincaré. Pour montrer qu'il est isotrope pour la forme d'intersection on peut se souvenir de l'argument suivant de Milnor [19]. Supposons que A et B soient des variétés différentiables compactes orientées immergées dans X, dont les bords sont contenus dans Σ. Alors les classes $\partial_*[A]$ et $\partial_*[B]$ sont dans l'image de $H_{2n}(X) \to H_{2n-1}(\Sigma)$, c'est-à-dire le noyau précédemment considéré. Supposons maintenant que, par une petite homotopie, on ait rendu A et B transverses. Alors $A \cap B$ est une sous-variété de dimension 1 de X, son bord $\partial A \cap \partial B$ est formé d'un nombre pair de points (fig. 2).
Le nombre d'intersection des classes $\partial_*[A]$ et $\partial_*[B]$ dans $H_{2n-1}(\Sigma)$ est alors le nombre de ces points, chacun étant compté avec son orientation. Puisque que l'on peut associer ces points deux à deux avec des orientations opposées, cette somme est nulle. Donc $\partial_*[A] \cdot_\Sigma \partial_*[B]$ est nul. Bien sûr, cet argument ne s'applique que dans le cas où toutes les classes intervenant sont représentées par des sous-variétés immergées.

Revenons à la situation du théorème de Novikov-Wall. On considère donc une variété (de dimension $4n$) obtenue en recollant les variétés M_1 et M_2 selon une sous-variété à bord, compacte, X_0, de leur bord. On désignera par Σ le bord de X_0 et par X_1 et X_2 le complémentaire de l'intérieur de X_0 dans les bords de M_1 et M_2. Ainsi, dans M, la sous-variété Σ peut être vue comme le bord de trois variétés de dimensions $4n-1$ distinctes : X_0, X_1, et X_2. Les images L_i des applications $H^{2n-1}(X_i) \to H^{2n-1}(\Sigma)$ fournissent donc trois lagrangiens de

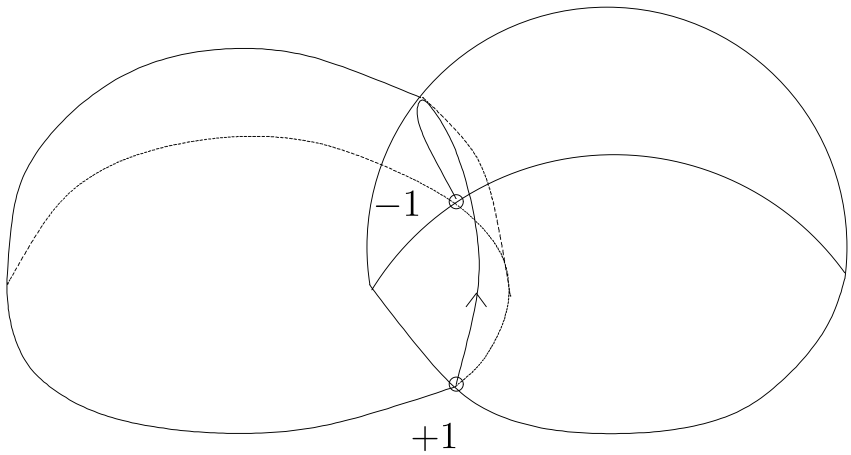

FIG. 2

$H^{2n-1}(\Sigma)$. On pourra calculer leur indice de Maslov dès lors qu'une orientation de Σ sera fixée.

Le début du paragraphe suivant est consacré à des précisions sur l'orientation relative de toutes les variétés considérées.

(3.2) Théorème de Novikov-Wall. L'homologie de M peut être calculée en considérant l'homologie du complexe $C_*(M_1 + M_2)$ formé des chaînes sommes de simplexes dont l'image est contenue ou bien dans M_1, ou bien dans M_2. Ainsi on dispose d'applications naturelles de $H_k(M, \partial M)$ vers $H_k(M_i, \partial M_i)$. On dispose pour la même raison d'applications de l'homologie de ∂M_i vers l'homologie relative de X_0 par exemple. De même, la cohomologie de M peut être calculée à partir du complexe $\mathrm{Hom}(C_*(M_1 + M_2), \mathbb{K})$.

On suppose qu'une classe fondamentale $[M] \in H_{4n}(M, \partial M)$ est fixée. Notons $[M_1]$ et $[M_2]$ les images de $[M]$ par les applications $H_{4n}(M, \partial M) \to H_{4n}(M_i, \partial M_i)$. Ce sont des classes fondamentales pour M_1 et M_2. Les classes $\partial_*[M_1]$ et $\partial_*[M_2]$ orientent ∂M_1 et ∂M_2 respectivement. On notera $[X_0]_1$ et $[X_0]_2$ leurs images respectives par les applications

$$H_{4n}(\partial M_i) \to H_{4n}(X_0, \Sigma)\,.$$

On a supposé que l'homéomorphisme utilisé pour effectuer notre somme connexe inversait l'orientation, c'est-à-dire que $[X_0]_1 = -[X_0]_2$. On notera $[X_0] = [X_0]_1$. Enfin on choisira $\partial_*[X_0] = [\Sigma]$ comme classe fondamentale de Σ. On a alors :

THÉORÈME (3.2.1). *Les variétés M,M_1,M_2 et Σ étant orientées comme ci-dessus, on a l'égalité suivante dans le groupe de Witt de $\mathbb{K}$:*

$$\sigma(M) = \sigma(M_1) + \sigma(M_2) + \tau(L_1, L_0, L_2)\,.$$

Résumons maintenant la trame de la preuve. Soit N une variété de dimension $4n$ à bord ∂N. On a la suite exacte suivante, associée à la paire $(N, \partial N)$:

$$H^{2n-1}(\partial N) \longrightarrow H^{2n}(N, \partial N) \xrightarrow{\pi} H^{2n}(N) \xrightarrow{\varphi} H^{2n}(\partial N).$$

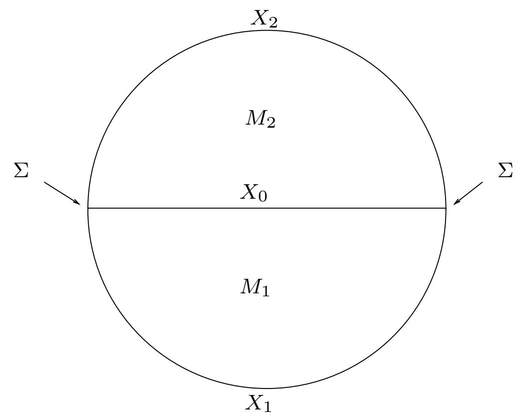

FIG. 3. Recollement le long d'une sous-variété à bord

La forme bilinéaire d'intersection $x \cdot_N y$, définie sur $H^{2n}(N) \times H^{2n}(N, \partial N)$, met les deux espaces du milieu en dualité. La signature de N sur $\mathbb{K}$ est par définition la classe dans le groupe de Witt de $\mathbb{K}$ de la forme bilinéaire symétrique $\mathcal{I}_N$ sur $H^{2n}(N, \partial N)$ définie par :

$$(x, y) \mapsto \pi(y) \cdot_N x.$$

Puisque $H^{2n}(N, \partial N)$ et $H^{2n}(N)$ sont duaux l'un de l'autre, on a que le radical de la forme $\mathcal{I}_N$ est égal à l'image de l'application $H^{2n-1}(\partial N) \to H^{2n}(N, \partial N)$. On choisit un supplémentaire G_N de cette image dans $H^{2n}(N, \partial N)$. On identifie ainsi les espaces $G_N \subset H^{2n}(N, \partial N)$ et $\operatorname{Ker} \varphi \subset H^{2n}(N)$. On applique cette remarque au cas où N est l'une des trois variétés M, M_1, M_2. On va alors écrire $G_M = G_{M_1} \oplus G_{M_2} \oplus V$, où la somme sera orthogonale relativement à $\mathcal{I}_M$ et où la restriction de $\mathcal{I}_M$ à G_{M_i} coïncidera avec $\mathcal{I}_{M_i}$. Le défaut δ recherché est alors :

$$\delta = \sigma(M) - \sigma(M_1) - \sigma(M_2) = \sigma(V).$$

On construira ensuite un sous-espace isotrope S de V. En notant $S^\perp$ l'orthogonal de S dans V et en écrivant $S^\perp = S \oplus L$, il suffira alors d'identifier $\sigma(L)$ à l'indice de Maslov, en vertu du Lemme (2.2.2) On peut maintenant rentrer dans les détails.

On a deux applications naturelles

$$H^{2n}(M_1, \partial M_1) \xrightarrow{f_1} H^{2n}(M)$$

$$H^{2n}(M_2, \partial M_1) \xrightarrow{f_2} H^{2n}(M).$$

Ces deux applications sont en fait définies au niveau des cochaînes. Par exemple, l'image par f_1 d'une cochaîne nulle sur le bord de M_1 est égal à son prolongement par 0 sur toutes les chaînes contenues dans M_2. Il est bien clair que f_1 et f_2 ont des images contenues dans $\operatorname{Ker} \varphi = \operatorname{Im} \pi$. On peut donc les considérer comme étant à valeurs dans G_M.

LEMME (3.2.2). *On a*

$$\mathcal{I}_{M_1}(u, v) = \mathcal{I}_M(f_1(u), f_1(v)) \text{ pour } u, v \in H^{2n}(M_1, \partial M_1),$$

$$\mathcal{I}_{M_2}(u,v) = \mathcal{I}_M(f_2(u), f_2(v)) \ \ \textit{pour } u,v \in H^{2n}(M_2, \partial M_2),$$
$$\mathcal{I}_M(f_1(u), f_2(v)) = 0 \ \ \textit{pour } u \in H^{2n}(M_1, \partial M_1) \textit{ et } v \in H^{2n}(M_2, \partial M_2)\,.$$

Preuve : on prouve le premier point seulement, les deux autres sont identiques. Quand on calcule le nombre $\mathcal{I}_M(f_1(v), f_1(u))$, on prend des cochaînes qui représentent chacune des deux classes $f_1(v)$ et $f_1(u)$, que l'on évalue sur tous les simplexes de la classe $[M]$. Il suffit de considérer les simplexes contenus dans M_1 puisque $f_1(u)$ et $f_1(v)$ sont nuls ailleurs. On évalue donc en fait le cup-produit des deux cochaînes sur $[M_1]$. □

Puisque les formes $\mathcal{I}_{M_1}$ et $\mathcal{I}_{M_2}$ sont non-dégénérées sur G_{M_1} et G_{M_2}, le lemme implique que f_1 et f_2 sont injectives, une fois restreintes à G_{M_1} et G_{M_2}. En outre puisque leurs images sont orthogonales on peut écrire :

$$G_M = G_{M_1} \oplus G_{M_2} \oplus V$$

où V est l'orthogonal de $G_{M_1} \oplus G_{M_2}$ dans G_M.

On peut maintenant considérer l'application $g_1 : H^{2n-1}(\partial M_1) \to H^{2n}(M_1, \partial M_1)$, issue de la suite exacte longue de cohomologie associée à la paire $(M_1, \partial M_1)$. En composant avec f_1 on obtient une application $s_1 : H^{2n-1}(\partial M_1) \to G_M$. On construit de même une application $s_2 : H^{2n-1}(\partial M_2) \to G_M$. On note S la somme des images de s_1 et s_2 dans G_M. Considérons un élément u de $H^{2n-1}(\partial M_1)$. Par le lemme précédent, et puisque $s_1 = f_1 g_1$, on a :

$$(*) \ \ \mathcal{I}_M(s_1(u), f_1(v)) = \mathcal{I}_{M_1}(g_1(u), v)$$

pour tout v de $H^{2n}(M_1, \partial M_1)$. Puisque l'image de g_1 est le radical de $\mathcal{I}_{M_1}$, cette quantité est toujours nulle. On en déduit que l'image de s_1 est orthogonale à l'image de G_{M_1} dans G_M. Puisque l'image de s_1 est contenue dans celle de f_1, elle est aussi orthogonale à G_{M_2}. On a de la même manière que l'image de s_2 est orthogonale à $G_{M_1} \oplus G_{M_2} \subset G_M$. Donc le sous-espace S est contenu dans V. En outre l'équation $(*)$, ainsi que l'équation correspondante pour s_2 assurent que S est isotrope. Il nous reste à calculer l'orthogonal de S dans V. Or l'orthogonal de S dans V est égal à l'orthogonal de $\operatorname{Im} f_1 + \operatorname{Im} f_2$ dans G_M.

Considérons la partie suivante de la suite exacte de Mayer-Vietoris :

$$H^{2n-1}(X_0) \longrightarrow H^{2n}(M) \longrightarrow H^{2n}(M_1) \oplus H^{2n}(M_2).$$

On note r_1 et r_2 les deux composantes de la flèche de droite dans le diagramme ci-dessus.

Proposition (3.2.3). *L'orthogonal de S dans V est :*

$$\operatorname{Im}(H^{2n-1}(X_0) \to H^{2n}(M)) \cap \operatorname{Ker} \varphi.$$

Preuve : Soit v un élément de $H^{2n}(M) \cap \operatorname{Ker} \varphi$. Il suffit d'établir les deux égalités :

$$\mathcal{I}_M(v, f_1(u)) = r_1(v) \cdot_{M_1} u$$
$$\mathcal{I}_M(v, f_2(u')) = r_2(v) \cdot_{M_2} u'$$

pour u dans $H^{2n}(M_1, \partial M_1)$ et u' dans $H^{2n}(M_2, \partial M_2)$. Puisque les formes bilinéaires $x \cdot_{M_i} y : H^{2n}(M_i) \times H^{2n}(M_i, \partial M_i) \to \mathbb{K}$ sont non-dégénérées on aura le résultat voulu. □

Il nous reste pour conclure à construire une application

$$\Upsilon : S^{\perp} \to W = \frac{L_0 \cap (L_1 + L_2)}{L_0 \cap L_1 + L_0 \cap L_2}$$

de noyau S, et envoyant la restriction de $\mathcal{I}_M$ à $S^{\perp}$ sur la forme définie à la fin du paragraphe 2.3 :

$$[x_0 = x_1 + x_2] \mapsto \omega(x_0, x_1)\,.$$

Regardons le diagramme commutatif suivant. Les lignes proviennent de la suite exacte de Mayer-Vietoris appliquée à M et ∂M décomposées respectivement selon M_1 et M_2 et X_1 et X_2.

$$\begin{array}{ccccc}
H^{2n-1}(M_1) \oplus H^{2n-1}(M_2) & \longrightarrow & H^{2n-1}(X_0) & \xrightarrow{\tau_1} & H^{2n}(M) \\
\downarrow & & \downarrow & & \downarrow \varphi \\
H^{2n-1}(X_1) \oplus H^{2n-1}(X_2) & \xrightarrow{\tau_2} & H^{2n-1}(\Sigma) & \xrightarrow{\tau_3} & H^{2n}(\partial M)
\end{array}$$

Si x_i est dans $H^{2n-1}(X_i)$, on notera $\bar{x}_i$ son image dans $H^{2n-1}(\Sigma)$.

Considérons un élément y de $S^{\perp}$. On peut écrire $y = \tau_1(x_0)$. Puisque $\varphi(y) = 0$, $\bar{x}_0$ est dans l'image de τ_2, c'est-à-dire dans $L_1 + L_2$. Puisqu'un élément du noyau de τ_1 s'envoie dans $L_0 \cap L_1 + L_0 \cap L_2$, on obtient une application Υ bien définie de $S^{\perp}$ dans W. Elle est surjective car si $\bar{x}_0 = \bar{x}_1 + \bar{x}_2$ représente un élément de W, $\tau_3(\bar{x}_0) = 0$ donc $\tau_1(x_0)$ est bien dans le noyau de φ. On a alors $\Upsilon(\tau_1(x_0)) = [\bar{x}_0]$. Vérifions que le noyau de Υ est S.

$$\begin{array}{ccccc}
 & & H^{2n-1}(\partial M_1) & \xrightarrow{s_1} & H^{2n}(M) \\
 & \swarrow & & \searrow & \uparrow \tau_1 \\
H^{2n-1}(X_1) & & & & H^{2n-1}(X_0) \\
 & \searrow & & \swarrow & \\
 & & H^{2n-1}(\Sigma) & &
\end{array}$$

Si x est dans $H^{2n-1}(\partial M_1)$, notons x_0 et x_1 ses restrictions à X_0 et X_1 respectivement. Puisque l'image de x_0 par τ_1 est $s_1(x)$, $\Upsilon(s_1(x))$ est représenté par $\bar{x}_0$. Mais $\bar{x}_0 = \bar{x}_1$ est dans $L_0 \cap L_1$. Donc l'image de s_1 est contenue dans le noyau de Υ. De même, l'image de s_2 est contenue dans ce noyau. Réciproquement, considérons un élément $y = \tau_1(x_0) \in S^{\perp}$ qui est dans le noyau de Υ. On peut écrire $\bar{x}_0 = \bar{a}_0 + \bar{b}_0$ avec $\bar{a}_0 \in L_0 \cap L_1$ et $\bar{b}_0 \in L_0 \cap L_1$. Ceci implique que l'on peut trouver $a \in H^{2n-1}(\partial M_1)$ et $b \in H^{2n-1}(\partial M_2)$, qui se restreignent à a_0 et b_0 sur X_0. Alors $y - s_1(a) - s_2(b) = \tau_1(v)$ où $v = x_0 - a_0 - b_0$ est dans l'image de l'application $H^{2n-1}(X_0, \Sigma) \to H^{2n-1}(X_0)$. On peut donc représenter v par un cocycle c sur X_0, nul sur Σ. Prolongeons c par 0 sur ∂M_1. Puisque la cohomologie de ∂M_1 peut être calculée à partir du complexe $\mathrm{Hom}(C_*(X_0 + X_1), \mathbb{K})$ où $C_*(X_0 + X_1)$ est le complexe des chaînes sommes de simplexes dans X_0 ou dans X_1, c est un cocycle. On vérifie facilement que $\tau_1(v) = s_1([c])$, donc $y = s_1(a) + s_2(b) + s_1([c])$ est dans S.

Identifions maintenant les formes quadratiques. Supposons que $\bar{x}_0 = \bar{x}_1 + \bar{x}_2$ soit dans $H^{2n-1}(\Sigma)$. On peut toujours trouver des cocycles φ_0,φ_1,φ_2 représentant x_0,x_1,x_2 teles que $\varphi_0 = \varphi_1 + \varphi_2$ sur Σ. Rappelons la construction de $\tau_1(x_0)$. On choisit des cochaînes ψ_1 et ψ_2, définies sur M_1 et M_2 respectivement, telles que

$\psi_1 - \psi_2 = \varphi_0$, sur X_0. On peut toujours supposer que $\varphi_1 = \psi_1$ sur X_1 et $\varphi_2 = -\psi_2$ sur X_2. La classe de cohomologie $\tau_1(x_0)$ est alors représentée par le cocycle ψ, égal à $\psi_1 \circ \partial$ sur M_1 et à $\psi_2 \circ \partial$ sur M_2. On doit vérifier l'égalité suivante :

$$\mathcal{I}_M(\tau_1(x_0), \tau_1(x_0)) = -\bar{x}_0 \cdot_\Sigma \bar{x}_1 \,.$$

Mais

$$\begin{aligned}\mathcal{I}_M(\tau_1(x_0), \tau_1(x_0)) &= (\psi \cup \psi)([M]) = (\psi_1 \circ \partial \cup \psi_1 \circ \partial)([M_1]) + (\psi_2 \circ \partial \cup \psi_2 \circ \partial)([M_2]) \\ &= (\psi_1 \cup \psi_1 \circ \partial)([X_1] + [X_0]) + (\psi_2 \cup \psi_2 \circ \partial)([X_2] - [X_0]) \\ &= ((\psi_1 - \psi_2) \cup \psi_1 \circ \partial)([X_0]) + (\psi_1 \cup \psi_1 \circ \partial)([X_1]) + (\psi_2 \cup \psi_2 \circ \partial)([X_2]).\end{aligned}$$

Le premier de ces trois termes vaut $x_0 \cdot_{X_0} j(\bar{x}_1)$. Ici j désigne l'application naturelle $H^{2n-1}(\Sigma) \to H^{2n}(X_0, \Sigma)$. Mais par le Lemme (3.1.1), ceci vaut $-\bar{x}_0 \cdot_\Sigma \bar{x}_1$. Il nous reste donc à vérifier que les deux derniers termes ci-dessus sont nuls. Mais $\psi_i \circ \partial$ est nul sur X_i, si on a supposé $\psi_i = \pm\varphi_i$ sur X_i.

(3.3) Cocycle signature de Meyer. Le théorème de Novikov-Wall permet par exemple de calculer la signature d'une variété différentiable de dimension 4 fibrée en surfaces au-dessus d'un pantalon (et donc de toute variété de dimension 4 fibrée au-dessus d'une surface). Le résultat obtenu a été prouvé initialement par Meyer [17] par des arguments de suites spectrales. La preuve suivante est tirée de [10]. Bien sûr, on peut se placer dans le cadre de la dimension $4n$.

On désigne par P la sphère S^2 privée de trois disques ouverts, avec son orientation usuelle. On note également F une variété (compacte sans bord) orientée de dimension $4n-2$. Si $\pi : M \to P$ est une fibration localement triviale au-dessus de P, de fibre F, dont le groupe structural préserve l'orientation de F, les orientations de F et de P induisent une orientation de M. Notons P_a et P_b les parties de P obtenues en découpant le long du chemin c (figure 4). Elles sont homéomorphes au produit d'un cercle par un intervalle. La composition $\pi^{-1}(P_a) \to P_a \to [0,1]$ est encore une fibration et est donc trivialisable. La variété $X_1 = \pi^{-1}(P_a)$ est homéomorphe à $N^{4n-1} \times [0,1]$, donc de signature nulle. Pour la même raison $X_2 = \pi^{-1}(P_b)$ est de signature nulle.

Notons X_0, l'image inverse par π du chemin c dans P, et Σ la réunion des fibres au-dessus de c_0 et de c_1. On identifie l'espace $H_{2n-1}(\Sigma) = H_{2n-1}(\pi^{-1}(c_1)) \oplus H_{2n-1}(\pi^{-1}(c_0))$ à $E = H_{2n-1}(\pi^{-1}(c_1) \oplus H_{2n-1}(\pi^{-1}(c_1))$ par la monodromie associée au chemin c. L'orientation de $\pi^{-1}(P_a)$ induit une orientation de X_0, qui induit elle-même une orientation de $\Sigma = \pi^{-1}(c_0) \cup \pi^{-1}(c_1)$. On note ω la forme symplectique sur $H_{2n-1}(\pi^{-1}(c_1))$ induite par cette orientation. En identifiant alors $H_{2n-1}(\pi^{-1}(c_0))$ à $H_{2n-1}(\pi^{-1}(c_1))$ par la monodromie de c, la forme symplectique fournie par l'orientation de $\pi^{-1}(c_0)$ s'envoie sur $-\omega$. Ainsi, l'espace $H_{2n-1}(\Sigma)$, orienté comme dans le théorème de Novikov-Wall, s'identifie à l'espace $E = H_{2n-1}(\pi^{-1}(c_1)) \oplus H_{2n-1}(\pi^{-1}(c_1))$ muni de la forme $\omega \oplus -\omega$.

Notons maintenant γ_a et γ_b les éléments de $\mathrm{Sp}(H_{2n-1}(\pi^{-1}(c_1)), \omega)$ correspondant à l'action des générateurs a et b du groupe fondamental du pantalon représentés sur la figure 4. Nous allons vérifier que les graphes de $-\mathrm{id}$, de $-\gamma_a$ et de $-\gamma_b$ dans E sont les noyaux des applications $H_{2n-1}(\Sigma, \mathbb{R}) \to H_{2n-1}(X_i, \mathbb{R})$. Prouvons le dans un cas seulement, les deux autres sont identiques. La monodromie le long de c fournit une application $f : \pi^{-1}(c_0) \times [0,1] \to X_0$ telle que f_t est un difféomorphisme de $\pi^{-1}(c_0)$ sur $\pi^{-1}(c_t)$, pour tout t. On peut alors construire

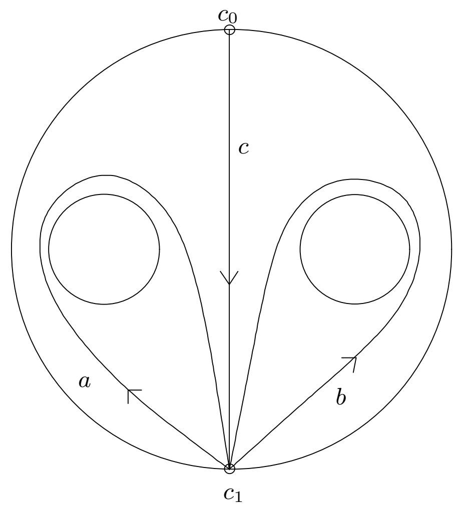

Fig. 4

un opérateur de prisme P, des k-chaînes de $\pi^{-1}(c_0)$ vers les $(k+1)$-chaînes de X_0, qui satisfait : $\partial P + P\partial = f_{1*} - f_{0*}$. Si u est un $(2n-1)$-cycle dans $\pi^{-1}(c_0)$, la classe définie par $(f_{1*}(u), -u)$ dans $H_{2n-1}(\Sigma)$ borde donc dans X_0. La classe correspondante dans E est $(f_{1*}(u), -f_{1*}(u))$. Le graphe de $-\mathrm{id}$ est donc bien le noyau de l'application $H_{2n-1}(\Sigma, \mathbb{R}) \to H_{2n-1}(X_0, \mathbb{R})$.

Définissons maintenant le cocycle de Meyer sur le groupe symplectique. On considère $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ muni de la forme symplectique $\omega \oplus -\omega$ (ω désignant la forme symplectique canonique de $\mathbb{R}^{2n}$). Si γ est un élément du groupe symplectique, son graphe

$$\mathrm{Graphe}(\gamma) = \{(v, \gamma(v)), v \in \mathbb{R}^{2n}\}$$

est un lagrangien de $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. Si γ_1,γ_2,γ_3 sont trois éléments du groupe symplectique, on peut donc considérer l'indice de Maslov de leurs graphes dans $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. On note $\mathrm{meyer}(\gamma_1, \gamma_2, \gamma_3)$ cet entier. Si x et y sont des éléments de $\mathrm{Sp}(2n, \mathbb{R})$, l'application $(u, v) \mapsto (x^{-1}(u), y(v))$ est une transformation symplectique de $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. On en déduit que $\mathrm{meyer}(\gamma_1, \gamma_2, \gamma_3) = \mathrm{meyer}(x\gamma_1 y, x\gamma_2 y, x\gamma_3 y)$. On pose alors

$$\mathrm{Meyer}(\gamma_1, \gamma_2) = \mathrm{meyer}(1, \gamma_1, \gamma_1\gamma_2)\,.$$

Ceci définit un 2-cocycle borné sur le groupe symplectique. En outre, Meyer est invariant par conjugaison. On renvoie à [3] pour le lien entre la classe de cohomologie définie par le cocycle de Meyer et la classe de cohomologie définie précédemment grâce à l'indice de Maslov, et pour un calcul détaillé du cocycle de Meyer sur $\mathrm{Sp}(2n, \mathbb{R})$.

Après avoir choisi une identification entre $H_{2n-1}(\pi^{-1}(c_1))$ et $\mathbb{R}^{2n}$, munis de leurs formes symplectiques respectives, on peut considérer γ_a et γ_b comme des éléments du groupe $\mathrm{Sp}(2n, \mathbb{R})$ (définis à conjugaison près). La signature de M est donc

$$\mathrm{meyer}(-\gamma_a, -\mathrm{id}, -\gamma_b) = \mathrm{meyer}(\mathrm{id}, \gamma_a^{-1}, \gamma_a^{-1}\gamma_b).$$

On obtient bien le théorème déjà prouvé par Meyer dans [17].

Théorème (3.3.1). *La signature de M ne dépend que de l'action des générateurs du groupe fondamental de P sur l'homologie de dimension $2n-1$ de la fibre. Avec les notations précédentes :*

$$\sigma(M) = \mathrm{Meyer}(\gamma_a^{-1}, \gamma_b).$$

(3.4) Signature et fonction de Morse. On suppose ici que M est une variété différentiable, compacte sans bord, de dimension $4n$. Considérons une fonction de Morse $f : M \to \mathbb{R}$ de points critiques $x_1, \ldots, x_p$. On suppose en outre que les valeurs critiques $\lambda_i = f(x_i)$ sont toutes distinctes, par exemple que $\lambda_1 < \cdots < \lambda_p$. Le théorème de Novikov-Wall permet alors de calculer la signature de M à partir de f. L'énoncé du théorème suivant m'a été suggéré par Étienne Ghys.

Théorème (3.4.1). *La signature de M s'écrit :*

$$\sigma(M) = \sum_i s(x_i)$$

où la somme porte sur les points critiques de f d'indice $2n$. L'entier $s(x_i)$ est compris entre -1 et 1 et est la signature de la variété $f^{-1}([\lambda_i - \epsilon, \lambda_i + \epsilon])$, pour tout ϵ suffisamment petit pour que l'intervalle $[\lambda_i - \epsilon, \lambda_i + \epsilon]$ ne contienne pas d'autre valeur critique que λ_i.

On décompose l'image de f dans $\mathbb{R}$ en une réunion d'intervalles compacts d'intérieurs disjoints, dont les extrémités ne sont pas des valeurs critiques (excepté pour le maximum et le minimum de f). On demande en outre que chacun de ces intervalles I contienne au plus une valeur critique. On va alors étudier la signature des variétés à bord $f^{-1}(I)$. Puisque chacun de ces intervalles est de la forme $I = [a, b]$ où a et b sont des valeurs régulières de f (à l'exception des deux intervalles contenant les extrémités de l'image de f), les variétés $f^{-1}(I)$ sont recollées les unes aux autres le long de niveaux $f^{-1}(a)$, qui sont des variétés compactes sans bord de dimension $4n - 1$. Ainsi par le théorème de Novikov usuel, la signature de M est la somme des signatures des $f^{-1}(I)$. Si l'intervalle I ne contient pas de valeur critique, $f^{-1}(I)$ est difféomorphe au produit $f^{-1}(a) \times [a, b]$ (par un résultat élémentaire de théorie de Morse). Sa signature est donc nulle. Il suffit donc de prendre en compte les intervalles I contenant une valeur critique. Ceux-ci peuvent être pris de la forme $f^{-1}([\lambda_i - \epsilon, \lambda_i + \epsilon])$ avec ϵ aussi petit qu'on le souhaite. En effet, changer ϵ en $\epsilon' > \epsilon$ revient à ajouter les variétés $f^{-1}([\lambda_i - \epsilon', \lambda_i - \epsilon])$ et $f^{-1}([\lambda_i + \epsilon, \lambda_i + \epsilon'])$ qui sont de signatures nulles. Par le théorème de Novikov usuel la signature ne s'en trouve pas affectée. On définit donc l'entier $s(x_i)$ comme la signature de $f^{-1}([\lambda_i - \epsilon, \lambda_i + \epsilon])$ (avec ϵ assez petit). Nous allons voir que cet entier est nul si le point critique n'est pas d'indice $2n$.

Notons donc l l'indice du point critique x_i. D'après la théorie de Morse, on peut trouver un plongement $\phi : S^{l-1} \times D^{4n-l} \to f^{-1}(\lambda_i - \epsilon/2)$, tel que la variété $f^{-1}([\lambda_i - \epsilon, \lambda_i + \epsilon])$ soit difféomorphe à la variété $f^{-1}([\lambda_i - \epsilon, \lambda_i - \epsilon/2])$, à laquelle on a recollé l'anse $D^l \times D^{4n-l}$, via le plongement ϕ. La variété $f^{-1}([\lambda_i - \epsilon, \lambda_i - \epsilon/2])$ est de signature nulle car est le produit d'une variété fermée de dimension $4n - 1$ par un intervalle. L'anse $D^l \times D^{4n-l}$ est également de signature nulle car est homéomorphe à une boule. Par le théorème de Novikov-Wall, la signature de $f^{-1}([\lambda_i - \epsilon, \lambda_i + \epsilon])$ est donc l'indice de Maslov de trois lagrangiens dans

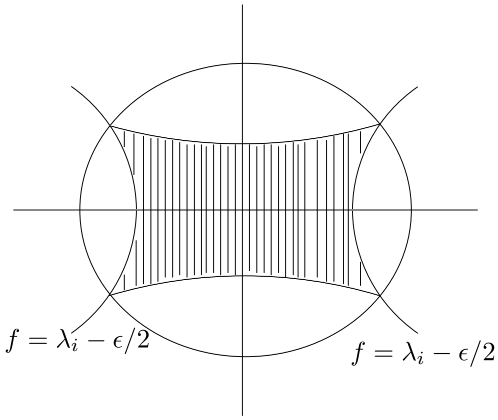

FIG. 5. La variété M_1

l'homologie de dimension $2n-1$ du bord de la sous-variété sur laquelle on a effectué le recollement, c'est-à-dire $S^{l-1} \times S^{4n-l-1}$. Or, si l est différent de $2n$, $S^{l-1} \times S^{4n-l-1}$ n'a pas d'homologie en dimension moitié. L'indice de Maslov cherché est donc nul, et l'entier $s(x_i)$ est nul également.

Supposons maintenant que l soit égal à $2n$. On suppose pour fixer les orientations que $D^{2n} \times D^{2n}$ est la variété M_1 dans le théorème de Novikov-Wall. On choisit une base (u, v) de l'homologie de dimension $2n-1$ de $S^{2n-1} \times S^{2n-1}$, telle que u corresponde à une classe fondamentale du premier facteur et v à une classe fondamentale du second facteur. En outre, on peut orienter $S^{2n-1} \times S^{2n-1}$, de sorte que $\mathcal{I}(u, v) = 1$, où $\mathcal{I}$ désigne le nombre d'intersection. On peut toujours supposer que le plongement ϕ envoie l'orientation choisie de $S^{2n-1} \times S^{2n-1}$ sur l'orientation qui lui est associée par le théorème de Novikov-Wall. Alors le lagrangien L_0 est la droite engendrée par v, le lagrangien L_1 est la droite engendrée par u. La droite L_2 est formée des cycles qui bordent dans le complémentaire dans $f^{-1}(\lambda_i - \epsilon/2)$ de l'image de ϕ. L'indice de Maslov vaut 0 si et seulement si la droite L_2 est égale à l'une des deux droites u ou v. Sinon, la droite L_2 est engendrée par un vecteur de la forme $\cos(\theta)u + \sin(\theta)v$ avec $\theta \in]0, \pi[$. L'indice de Maslov cherché est 1 si $\theta < \pi/2$ et -1 si $\theta > \pi/2$. On obtient bien dans tous les cas un indice borné par 1.

4. Grassmannienne Lagrangienne de $\mathbb{R}^{2n}$ et revêtement universel du groupe symplectique

Dans ce qui suit on notera (p_α, q_α) les coordonnées d'un point de $\mathbb{R}^{2n}$. On peut identifier $\mathbb{R}^{2n}$ à $\mathbb{C}^n$ par les coordonnées $z_\alpha = p_\alpha + iq_\alpha$. On désigne par ω la forme symplectique usuelle sur $\mathbb{R}^{2n}$:

$$\omega((p_\alpha, q_\alpha), (p'_\alpha, q'_\alpha)) = \sum_{\alpha=1}^{n} q'_\alpha p_\alpha - p'_\alpha q_\alpha$$

et par $<,>$ le produit scalaire euclidien :

$$< (p_\alpha, q_\alpha), (p'_\alpha, q'_\alpha) >= \sum_{\alpha=1}^{n} p_\alpha p'_\alpha + q_\alpha q'_\alpha.$$

En notant $(,)$ la forme hermitienne canonique sur $\mathbb{C}^n$, on a pour des vecteurs x et y de $\mathbb{C}^n$:

$$(x, y) =< x, y > -i\omega(x, y).$$

La forme $(,)$ est $\mathbb{C}$-linéaire à gauche. On désignera par $\mathbb{R}^n$ le sous-espace de $\mathbb{R}^{2n}$ d'équation $q = 0$, et par $i\mathbb{R}^n$ le sous-espace d'équation $p = 0$.

Les constructions et les résultats des paragraphes 4.1 et 4.2 sont dus à Arnold et proviennent des articles [1] et [2].

(4.1) Propriétés de la Grassmannienne Lagrangienne. On notera Λ_n l'ensemble des sous-espaces lagrangiens de $\mathbb{R}^{2n}$. Le groupe unitaire $U(n)$ est contenu dans le groupe symplectique $\mathrm{Sp}(2n, \mathbb{R})$. Si L est un lagrangien de $\mathbb{R}^{2n}$, on peut trouver une base $v_1, \dots, v_n$ de L, orthonormée pour le produit scalaire euclidien. Puisque L est isotrope pour ω, la relation entre les structures complexe, euclidienne et symplectique, assure que $\{v_l\}$ est une base orthonormée de $\mathbb{C}^n$ pour sa structure hermitienne. Donc L est l'image de $\mathbb{R}^n$ par une transformation unitaire. L'action du groupe unitaire sur Λ_n est donc transitive, et puisque le stabilisateur de $\mathbb{R}^n$ dans $U(n)$ est $O(n)$, on identifie Λ_n à $U(n)/O(n)$. Soit $\det^2 : \Lambda_n \to S^1$, l'application qui envoie l'élément $u \cdot O(n)$ sur le carré du déterminant de u. On a alors un diagramme commutatif :

$$\begin{array}{ccccc}
SO(n) & \longrightarrow & O(n) & \longrightarrow & S^0 \\
\downarrow & & \downarrow & & \downarrow \\
SU(n) & \longrightarrow & U(n) & \longrightarrow & S^1 \\
\downarrow & & \downarrow & & \downarrow \\
SU(n)/SO(n) & \longrightarrow & U(n)/O(n) & \longrightarrow & S^1
\end{array}$$

Les deux premières lignes horizontales sont réalisées par le déterminant, la troisième par le carré du déterminant. La flèche verticale de droite est fournie par le revêtement du cercle $z \mapsto z^2$. Toutes les suites qui apparaissent sont des fibrations. En appliquant deux fois la suite exacte d'homotopie associée à une fibration, sur la colonne de gauche puis sur la ligne du bas, on obtient que $SU(n)/SO(n)$ est simplement connexe, puis que l'application :

$$\det^2 : \Lambda_n \to S^1$$

induit un isomorphisme entre les groupes fondamentaux. On en déduit que

$$H^1(\Lambda_n, \mathbb{Z}) = \mathrm{Hom}(\pi_1(\Lambda_n), \mathbb{Z})$$

est isomorphe à $\mathbb{Z}$.

Notations. On notera $\tilde{\Lambda}_n$ le revêtement universel de Λ_n. On choisit un point base $\tilde{*}$ dans $\tilde{\Lambda}_n$, et on note $*$ son image dans Λ_n. Ceci permet d'identifier le groupe $\pi_1(\Lambda_n, *)$ au groupe des automorphismes du revêtement cyclique $\pi : \tilde{\Lambda}_n \to \Lambda_n$. On notera T le générateur de ce groupe correspondant à un générateur de $\pi_1(S^1)$ avec l'orientation habituelle du cercle, Γ le groupe des homéomorphismes de $\tilde{\Lambda}_n$

qui relèvent l'action d'un élément du groupe symplectique sur Λ_n, et $\mathrm{PSp}(2n, \mathbb{R})$ le quotient du groupe symplectique par son centre $\{\pm \mathrm{Id}\}$.

Enfin si λ est un lagrangien on notera $\mathcal{O}(\lambda)$ l'ouvert des lagrangiens qui lui sont transverses, et Λ_n^k l'ensemble des lagrangiens dont l'intersection avec $\mathbb{R}^n$ est de dimension k. Si K est une partie de $\{1, \cdots, n\}$, on note λ_K le lagrangien défini par les équations

$$p_j = 0,\ j \in K$$
$$q_j = 0,\ j \notin K\,.$$

C'est l'image de $\mathbb{R}^n$ par la transformation unitaire I_K :

$$I_K(z_j) = iz_j,\ j \in K$$
$$I_K(z_j) = z_j,\ j \notin K.$$

On notera E la matrice identité, et I la multiplication par i, c'est-à-dire $I = I_{\{1,\cdots,n\}}$.

Lemme (4.1.1). *Si $\lambda \in \Lambda_n^k$, alors il existe une partie K de $\{1, \cdots, n\}$ à k éléments telle que λ soit transverse à λ_K.*

Preuve : si $\lambda_0 = \lambda \cap \mathbb{R}^n$, il est clair que l'on peut trouver K de cardinal k tel que λ_0 et $\lambda_K \cap \mathbb{R}^n$ soient supplémentaires dans $\mathbb{R}^n$. Alors puisque λ et λ_K sont lagrangiens et que $\mathbb{R}^n = \lambda_0 \oplus \lambda_K \cap \mathbb{R}^n$:

$$\omega(\lambda \cap \lambda_K, \mathbb{R}^n) = 0.$$

Donc $\lambda \cap \lambda_K$ est contenu dans $\mathbb{R}^n$ donc nul car λ_0 et $\lambda_K \cap \mathbb{R}^n$ sont supplémentaires. □

L'ouvert des lagrangiens transverses à $\mathbb{R}^n$ est paramétré par l'ensemble des matrices symétriques réelles de taille $n \times n$. A la matrice S on associe le plan lagrangien λ_S défini par l'équation $p = S(q)$. L'ouvert $\mathcal{O}(\lambda_K)$ est donc paramétré par les matrices symétriques réelles de la manière suivante : à la matrice S on associe le plan $I_K(\lambda_S)$. On obtient ainsi un atlas de la grassmannienne lagrangienne à 2^n cartes. On peut bien sûr construire un système de coordonnées identique en utilisant un lagrangien quelconque à la place de $\mathbb{R}^n$, une structure complexe compatible avec ω quelconque, et une base orthonormée de ce lagrangien. Nous utiliserons ce fait par la suite.

(4.2) Définition géométrique de l'indice de Maslov. Si λ est un lagrangien, nous noterons $t(\lambda)$ l'ensemble des lagrangiens qui ne lui sont pas transverses, que nous appellerons la *traîne* de λ, en suivant la terminologie de Arnold. Nous allons voir que $t(\lambda)$ est une sous-variété stratifiée de Λ_n. Avec les notations précédentes on a $t(\mathbb{R}^n) = \cup_{k\geq 1} \Lambda_n^k$. L'ensemble Λ_n^k est une sous-variété de Λ_n. En effet, supposons que le lagrangien λ_0 ait une intersection avec $\mathbb{R}^n$ de dimension k. Par le lemme 4.1, on peut écrire $\lambda_0 = I_K(\lambda_{S_0})$ avec K de cardinal k. L'application qui au vecteur q de $\mathbb{R}^n$ associe $\varphi(q) = I_K(Sq, q)$ est un isomorphisme de $\mathbb{R}^n$ sur $I_K(\lambda_S)$. Dire que $\varphi(q)$ est dans $\mathbb{R}^n$ équivaut à :

$$\begin{cases} q_j = 0,\ j \notin K \\ \sum_{s\in K} S_{rs} q_s = 0,\ r \in K. \end{cases}$$

Donc l'intersection $I_K(\lambda_S) \cap \mathbb{R}^n$ est de dimension k si et seulement si $S_{rs} = 0$ pour tous les couples (r, s) de $K \times K$. On en déduit que Λ_n^k est une sous-variété de

Λ_n, de codimension $k(k+1)/2$. On peut aussi remarquer que dans cette carte, le fait d'être transverse à $\mathbb{R}^n$ se traduit par la condition $\det(S_{K\times K}) \neq 0$ (où $S_{K\times K}$ est la matrice extraite de S obtenue en ne conservant que les indices de K). Considérons le flot $\phi^t(\lambda) = e^{tI}(\lambda)$ sur Λ_n. Notons v le champ de vecteurs associé :

$$v(\lambda) = \frac{d}{dt}_{t=0}(e^{tI}(\lambda))\,.$$

Supposons que $\lambda \in \Lambda_n^k$ s'écrive $\lambda = I_K(\lambda_S)$ (avec K de cardinal $k \geq 1$). Nous allons calculer les coordonnées de v dans les cartes précédemment construites. Puisque e^{tI} et I_K commutent, il suffit de calculer les coordonnées du plan $e^{tI}(\lambda_S)$. Pour t assez petit on a $e^{tI}(\lambda_S) = \lambda_{S(t)}$ avec

$$S(t) = (\cos(t)S - \sin(t)E)(\sin(t)S + \cos(t)E)^{-1}.$$

On a $S'(0) = -(E+S^2)$, dont les coefficients diagonaux ne sont jamais nuls, donc $v(\lambda)$ n'est pas tangent à Λ_n^k.

On a donc construit un champ de vecteurs v qui n'est tangent à aucune strate de la traîne de $\mathbb{R}^n$. Puisque $U(n)$ agit transitivement sur la grassmannienne lagrangienne et commute avec ϕ^t, ce champ n'est tangent à aucune strate de $t(\alpha)$, quel que soit le lagrangien α. Ceci nous permet d'orienter transversalement la partie régulière (de codimension 1) de $t(\alpha)$.

On aurait pu faire la même construction avec le flot donné par l'action de e^{tC}, où C est une structure complexe compatible avec ω quelconque. Puisque l'ensemble de ces structures est contractile, l'orientation de la partie régulière de $t(\alpha)$ ne dépend pas du choix de C.

La traîne $t(\alpha)$ d'un lagrangien possède un point privilégié : son *sommet* α. Choisissons un lagrangien β transverse à α. L'ouvert $\mathcal{O}(\beta)$ est difféomorphe à l'ensemble des formes bilinéaires symétriques sur α de la manière suivante : un lagrangien L de $\mathcal{O}(\beta)$ s'écrit comme le graphe d'une application linéaire $f : \alpha \to \beta$. On lui associe la forme $\omega(x, f(y))$. L'ensemble $\mathcal{O}(\beta) \cap t(\alpha)$ est formé des formes dégénérées, et $\mathcal{O}(\beta) - t(\alpha)$ a $n+1$ composantes connexes paramétrées par la signature. Si C est une structure complexe compatible avec ω, pour t dans $]0,\pi[$, α et $e^{tC}(\alpha)$ sont transverses. Il n'est pas difficile de vérifier que, pour t assez petit $e^{tC}(\alpha)$ est dans la composante connexe de $\mathcal{O}(\beta) - t(\alpha)$ formée des formes définies positives (il suffit pour cela de choisir C de sorte que $\beta = C(\alpha)$, la forme associée au lagrangien $e^{tC}(\alpha)$ admet alors pour seule valeur propre $\tan(t)$). On y fera référence comme la composante privilégiée du voisinage du sommet de la traîne de α.

Soit α un lagrangien et c_t une courbe dans Λ_n dont les extrémités c_0 et c_1 sont transverses à α. On peut modifier la courbe c par une petite homotopie (à extrémités fixes) de telle sorte qu'elle ne rencontre $t(\alpha)$ que dans sa partie régulière (formée des lagrangiens dont l'intersection avec α est de dimension 1) et ce de manière transverse. Alors, grâce à l'orientation transverse de $t(\alpha)$ on peut compter algébriquement le nombre de ses points d'intersection avec la courbe c. On note $I(\alpha, c)$ cet entier. Si β est transverse à α, on obtient ainsi un morphisme $I(\alpha, -) : \pi_1(\Lambda_n, \beta) \to \mathbb{Z}$.

PROPOSITION (4.2.1). *L'élément $I(\alpha,-)$ est un générateur de $H^1(\Lambda_n, \mathbb{Z})$.*

Preuve : On peut supposer $\alpha = \mathbb{R}^n$. De plus cette propriété ne dépend pas du choix du point β (transverse à α), on peut donc supposer que $\beta = U(\alpha)$ où U est une matrice unitaire diagonale de coefficients $e^{ia_1}, \cdots, e^{ia_n}$ avec $-\pi < a_1 < \cdots < a_n < 0$. La courbe $c_t = e^{It}(\beta)$ $(t \in [0, \pi])$ rencontre n fois la traîne de α, dans sa partie régulière, et selon la direction positive. Donc $I(\alpha, c) = n$. Par ailleurs, il est clair que $\det^2(c)$ fait n tours sur le cercle quand t varie de 0 à π. □

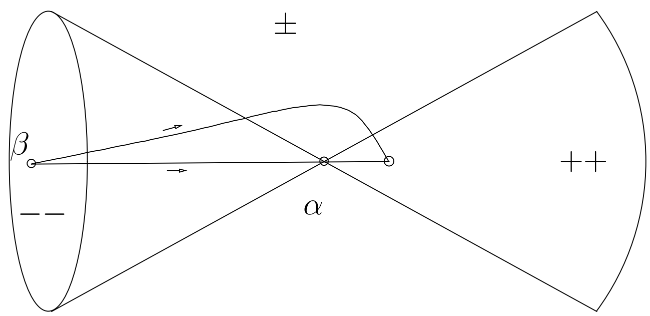

FIG. 6. Structure de la traîne au voisinage de son sommet dans Λ_2

Supposons maintenant que $\tilde{\alpha}$ et $\tilde{\beta}$ soient deux points de $\tilde{\Lambda}_n$, d'images α et β transverses dans Λ_n. Choisissons une courbe de $\tilde{\beta}$ à $\tilde{\alpha}$ dans $\tilde{\Lambda}_n$ et notons c sa projection dans Λ_n. On note $\hat{c}$ la courbe obtenue en prolongeant c par $e^{It}(\alpha)$ pour $t \in [0, \pi/2]$. La courbe $\hat{c}$ a des extrémités transverses à α, et sa classe d'homotopie ne dépend que de $\tilde{\alpha}$ et $\tilde{\beta}$ car $\tilde{\Lambda}_n$ est simplement connexe. On définit alors $m(\tilde{\alpha}, \tilde{\beta}) = I(\alpha, \hat{c})$. Cet entier ne dépend pas du choix de la structure complexe utilisée pour prolonger c.

Dans le cas où α et β ne sont pas transverses, on fait précéder la courbe c d'une courbe de la forme $e^{-tC}(\beta)$, $t \in [\epsilon, 0]$. Dans une carte construite comme précédemment, en faisant jouer à α le rôle de $\mathbb{R}^n$, et en choisissant C comme structure complexe, les coordonnées de $e^{-tC}(\beta)$ sont $S(t) = S + t(E + S^2) + O(t^2)$, si β admet S comme coordonnées. Puisque $E + S^2$ est définie positive, et $S_{K \times K} = 0$, $S(t)_{K \times K}$ est inversible pour t assez petit, c'est-à-dire $e^{-tC}(\beta)$ est transverse à α. On note encore $\hat{c}$ la courbe obtenue à partir de c en la modifiant comme indiqué pour que ses deux extrémités soient transverses à α et on définit $m(\tilde{\alpha}, \tilde{\beta}) = I(\alpha, \hat{c}) - \dim(\alpha \cap \beta)/2$.

PROPOSITION (4.2.2). *La fonction* $m : \tilde{\Lambda}_n \times \tilde{\Lambda}_n \to \mathbb{Z}$ *est* Γ*-invariante :*

$$m(\tilde{g}(\tilde{\alpha}), \tilde{g}(\tilde{\beta})) = m(\tilde{\alpha}, \tilde{\beta}),\ \tilde{g} \in \Gamma.$$

Preuve : choisissons deux points $\tilde{\alpha}$ et $\tilde{\beta}$ de $\tilde{\Lambda}_n$ et $\tilde{g}$ dans Γ, d'image g dans $\mathrm{PSp}(2n, \mathbb{R})$. Notons c la projection dans Λ_n d'une courbe $\tilde{c}$ de $\tilde{\beta}$ à $\tilde{\alpha}$, $\hat{c}$ la courbe obtenue à partir de c en la modifiant convenablement aux extrémités. Le terme $-\dim(\alpha \cap \beta)/2$ est clairement invariant par l'action de $\mathrm{PSp}(2n, \mathbb{R})$. Il reste donc à voir d'une part que $I(\alpha, \hat{c}) = I(g\alpha, g(\hat{c}))$ et d'autre part que la courbe $g(\hat{c})$ est homotope à une courbe obtenue à partir de $g(c)$ par le procédé que nous avons expliqué. Le second point résulte du fait que la courbe $e^{tC}(\alpha)$ s'envoie sur la courbe $e^{tgCg^{-1}}(g(\alpha))$. L'application gCg^{-1} est encore une structure complexe compatible avec ω. Le premier résulte du fait que l'action du groupe symplectique sur Λ_n préserve l'orientation des traînes. □

L'ouvert des lagrangiens transverses à α étant contractile, son image inverse par π, $\tilde{\Lambda}_n - \widetilde{t(\alpha)}$, est une réunion disjointe d'ouverts contractiles permutés par l'action de T. Chacun de ces ouverts est homéomorphe à $\Lambda_n - t(\alpha)$. La fonction $m(\tilde{\alpha}, -)$ est constante sur chacun de ces ouverts. Le théorème suivant a été prouvé par Arnold dans le cas où les lagrangiens considérés sont tous deux à deux transverses [2]. Des preuves complètes se trouvent dans [8] et [9].

Théorème (4.2.3). *La fonction m a les propriétés suivantes :*

1. $m(T(\tilde{\alpha}), \tilde{\beta}) = m(\tilde{\alpha}, \tilde{\beta}) + 1$.
2. $m(\tilde{\alpha}, \tilde{\beta}) + m(\tilde{\beta}, \tilde{\alpha}) = n$ *si* $\pi(\tilde{\alpha})$ *et* $\pi(\tilde{\beta})$ *sont transverses.*
3. $m_0(\tilde{\alpha}, \tilde{\beta}) + m_0(\tilde{\beta}, \tilde{\gamma}) - m_0(\tilde{\alpha}, \tilde{\gamma}) = \tau(\pi(\tilde{\alpha}), \pi(\tilde{\beta}), \pi(\tilde{\gamma}))$ *avec* $m_0 = n - 2m$.

Preuve : pour calculer $m(T(\tilde{\alpha}), \tilde{\beta})$ on considère un chemin c de β (ou d'un point proche de β et transverse à α construit comme précédemment si β rencontre α) à α. On le fait suivre d'un chemin d de α à α' de la forme $d(t) = e^{tC}(\alpha)$. On note encore T un générateur de $\pi_1(\Lambda_n, \alpha')$. On a $I(\alpha, c * d) = m(\tilde{\alpha}, \tilde{\beta})$ et $I(\alpha, c * d * T) = m(\tilde{\alpha}, \tilde{\beta}) + 1$ car $I(\alpha, -)$ engendre le groupe $H^1(\Lambda_n, \mathbb{Z})$. Mais $c * d * T$ est homotope à $c * (d * T * \bar{d}) * d$ et $c * (d * T * \bar{d})$ est l'image d'un chemin de $\tilde{\beta}$ à $T(\tilde{\alpha})$. Donc $I(\alpha, c * d * T) = m(T(\tilde{\alpha}), \tilde{\beta})$.

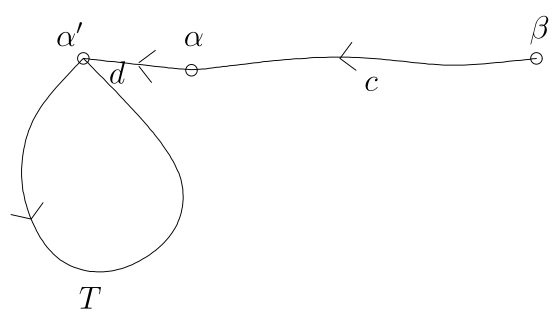

Fig. 7

Pour prouver le second point, on considère la fonction $m(\tilde{\alpha}, -) + m(-, \tilde{\alpha})$. Il suffit de prouver qu'elle est constante égale à n sur $\tilde{\Lambda}_n - \widetilde{t(\alpha)}$. On va prouver que c'est une constante $c_{\tilde{\alpha}}$. Alors l'invariance par Γ assurera que $c_{\tilde{\alpha}}$ ne dépend pas de $\tilde{\alpha}$. Il suffira de calculer cette constante sur un exemple.

Il est bien clair que $m(\tilde{\alpha}, -)$ est constante sur chaque composante connexe de $\tilde{\Lambda}_n - \widetilde{t(\alpha)}$. Par ailleurs on a $m(\tilde{g}(\tilde{\beta}), \tilde{\alpha}) = m(\tilde{\beta}, \tilde{g}^{-1}(\tilde{\alpha}))$ pour $\tilde{g} \in \Gamma$. Cet entier reste égal à $m(\tilde{\beta}, \tilde{\alpha})$ si $\tilde{g}$ reste dans un voisinage connexe V de 1 dans Γ tel que $\tilde{g}^{-1}(\tilde{\alpha})$ reste transverse à $\tilde{\beta}$ pour $\tilde{g}$ dans V. Mais $\tilde{g}(\tilde{\beta})$ décrit un voisinage de $\tilde{\beta}$ pour $\tilde{g}$ dans V. Donc $m(-, \tilde{\alpha})$ est localement constante. La fonction $m(\tilde{\alpha}, -) +$

$m(-,\tilde{\alpha})$ est donc localement constante sur $\tilde{\Lambda}_n - \widetilde{t(\alpha)}$. Par le premier point, elle est T-invariante donc constante.

Il nous reste à calculer cette constante sur un exemple simple. Considérons les lagrangiens $\mathbb{R}^n$ et $i(\mathbb{R}^n)$. On peut les joindre par le chemin c donné par : $e^{tI}(\mathbb{R}^n)$, $t \in [0, \frac{\pi}{2}]$. Pour calculer $m(i(\tilde{\mathbb{R}^n}), \tilde{\mathbb{R}^n})$ on prolonge c par le chemin $e^{tI}(i\mathbb{R}^n)$ ($t \in [0, \pi/2]$) à valeur dans la composante privilégié de $\tilde{\Lambda}_n - \widetilde{t(i(\mathbb{R}^n)}$. On doit donc calculer le nombre d'intersection de $e^{tI}(\mathbb{R}^n)$ ($t \in [0, \pi]$) avec la traîne de $i\mathbb{R}^n$. Cet entier vaut n par la Proposition (4.2.1). Il reste à voir que $m(\tilde{\mathbb{R}^n}, i(\tilde{\mathbb{R}^n}))$ vaut 0. Mais pour le calculer on considère le chemin $e^{(\frac{\pi}{2}-t)I}(\mathbb{R}^n)$ $t \in [0, \frac{\pi}{2}]$. En le prolongeant par $e^{tI}(\mathbb{R}^n)$ ($t \in [0, \pi/2]$) on obtient un chemin homotope à un chemin constant. Donc $m(\tilde{\mathbb{R}^n}, i(\tilde{\mathbb{R}^n})) = 0$.

Prouvons maintenant le dernier point. Il est bien clair que $m_0(\tilde{\alpha}, \tilde{\beta}) + m_0(\tilde{\beta}, \tilde{\gamma}) - m_0(\tilde{\alpha}, \tilde{\gamma})$ ne dépend que des images α,β,γ de $\tilde{\alpha}$,$\tilde{\beta}$,$\tilde{\gamma}$ dans Λ_n. On note $\mu(\alpha, \beta, \gamma)$ cette quantité. C'est un cocycle car son relevé à $\tilde{\Lambda}_n$ est un cobord. Pour prouver qu'il coïncide avec τ il suffit de le faire dans le cas où le troisième lagrangien est transverse aux deux autres. En effet supposons que τ et μ coïncident sur les triplets ayant cette propriété. Soit L_1,L_2,L_3 un triplet quelconque. On peut trouver un lagrangien L_4 qui soit transverse à L_1,L_2 et L_3 à la fois. Alors

$$\mu(L_1, L_2, L_3) = \mu(L_1, L_2, L_4) + \mu(L_2, L_3, L_4) - \mu(L_1, L_3, L_4)$$
$$\tau(L_1, L_2, L_3) = \tau(L_1, L_2, L_4) + \tau(L_2, L_3, L_4) - \tau(L_1, L_3, L_4)$$

car τ et μ vérifient chacun la relation de cocycle du Théorème (2.2.1). Les membres de droites coïncident, donc $\tau(L_1, L_2, L_3) = \mu(L_1, L_2, L_3)$.

Prouvons maintenant le résultat dans le cas où le troisième lagrangien est transverse aux deux autres. Puisque l'on a vu que l'on peut toujours utiliser une structure complexe compatible avec la structure symplectique on peut supposer que $\alpha = \mathbb{R}^n$, $\gamma = i\mathbb{R}^n$, et que β a pour équation $q = S_\beta p$ où S_β est une matrice symétrique. Alors, d'après la Proposition (2.3.1), l'indice de Maslov $\tau(\alpha, \beta, \gamma)$ est la signature de la forme quadratique $q(x) = \omega(x, iS_\beta x) = < x, S_\beta x >$ sur $\mathbb{R}^n$, c'est-à-dire la signature de S_β. Notons ν_+ le nombre de valeurs propres positives de S_β et ν_- le nombre de valeurs propres négatives. On choisit trois chemins reliant $\mathbb{R}^n$, $i\mathbb{R}^n$, β comme suit. On relie $\mathbb{R}^n$ à $i\mathbb{R}^n$ par le chemin de lagrangiens c_1 d'équation $(p, -tp)$ pour $t \in [0, \infty]$; β à $\mathbb{R}^n$ par le chemin c_2 d'équation $(p, (1-t)S_\beta p)$, $t \in [0, 1]$, et β à $i\mathbb{R}^n$ par le chemin c_3 d'équation $(p, S_\beta p - tp)$, $t \in [0, \infty]$. Il n'est pas difficile de voir que le lacet ainsi formé se relève dans $\tilde{\Lambda}_n$ en un lacet. On peut donc trouver trois relevés $\tilde{\mathbb{R}^n}$, $\tilde{i\mathbb{R}^n}$, $\tilde{\beta}$, tels que les classes d'homotopie de chemins déterminées par deux de ces points se projettent sur les chemins c_1,c_2 et c_3. De plus, en paramétrant $\mathcal{O}(i\mathbb{R}^n)$ par les coordonnées qui à une matrice symétrique S associent le plan d'équation $q = Sp$, le champ de vecteur v a pour coordonnées $v(S) = E + S^2$. Enfin, dans les coordonnées qui à la matrice symétrique S associent le lagrangien d'équation $q = Sp$, une matrice définie positive est dans la composante privilégiée pour $\mathbb{R}^n$, et une matrice inversible définie négative est dans la composante privilégiée pour $i\mathbb{R}^n$.

Le chemin c_1 est formé de lagrangiens transverses à $i\mathbb{R}^n$ et qui sont dans la composantes privilégiée pour $i\mathbb{R}^n$, au voisinage de $t = \infty$. Donc $m(i\tilde{\mathbb{R}^n}, \tilde{\mathbb{R}^n}) = 0$, et $m_0(i\tilde{\mathbb{R}^n}, \tilde{\mathbb{R}^n}) = n$. Le chemin c_3 est lui aussi formé de lagrangiens transverses à $i\mathbb{R}^n$ et arrive dans la composante privilégiée au voisinage de l'infini.

Donc $m_0(i\tilde{\mathbb{R}^n}, \tilde{\beta}) = n$. Quand on prolonge aux extrémités le chemin c_2 dans la direction du champ v, on obtient un chemin d'un lagrangien β' (de coordonnées une matrice $S_{\beta'}$) à un lagrangien proche de $\mathbb{R}^n$ (de coordonnées une matrice proche de 0 et définie positive). L'indice d'intersection de ce chemin avec $t(\mathbb{R}^n)$ est le nombre de valeurs propres négatives de $S_{\beta'}$ d'après le lemme suivant. Toujours d'après ce lemme ce nombre de valeurs propres négatives est $\nu_- + \dim(\beta \cap \mathbb{R}^n)$. Enfin $m_0(\tilde{\mathbb{R}^n}, \tilde{\beta}) = n - 2\nu_- - 2\dim(\beta \cap \mathbb{R}^n) + \dim(\beta \cap \mathbb{R}^n)$. Donc $\mu(\mathbb{R}^n, \beta, i\mathbb{R}^n) = n - 2\nu_- - \dim(\beta \cap \mathbb{R}^n) - n + n = \nu_+ - \nu_-$ [car rang $S_{\beta'} = n = \nu_+ + \nu_- + \dim(\beta \cap \mathbb{R}^n)$]. □

LEMME (4.2.4). *Soit $s \in [-\epsilon, \epsilon] \mapsto M_s$ un chemin lisse de matrices symétriques. On suppose que M_s est non-dégénérée pour s non-nul, et que M'_s est définie positive sur le noyau de M_0. Alors le nombre de valeurs propres positives de M_ϵ est égal au nombre de valeurs propres positives de $M_{-\epsilon}$, augmenté de la dimension du noyau de M_0.*

Preuve : on peut supposer que

$$M_s = \begin{pmatrix} A_s & B_s \\ B_s^t & C_s \end{pmatrix}$$

avec C_0 inversible, et A_0 et B_0 nuls ($C_s \in GL_k(\mathbb{R})$ où k est le rang de M_0). Alors

$$M_s = \begin{pmatrix} 1 & B_s C_s^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_s - B_s C_s^{-1} B_s^t & 0 \\ 0 & C_s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_s^{-1} B_s^t & 1 \end{pmatrix}.$$

Donc M_s a même signature que la matrice

$$\begin{pmatrix} A_s - B_s C_s^{-1} B_s^t & 0 \\ 0 & C_s \end{pmatrix} = \begin{pmatrix} sA'_0 + O(s^2) & 0 \\ 0 & C_s \end{pmatrix}.$$

Par hypothèse A'_0 est définie positive. On en déduit le résultat. □

Avant d'en finir avec la définition géométrique de l'indice de Maslov, on peut décrire ce qu'est l'indice de Maslov dans $\mathbb{R}^2$. Toute droite du plan est alors lagrangienne. La grassmannienne lagrangienne s'identifie donc à la droite projective. L'application $\det^2$ précédemment rencontrée est une bijection. En utilisant l'orientation usuelle de la droite projective on peut décrire l'indice de Maslov de trois droites D_1,D_2,D_3. Si deux d'entre elles sont confondues $\tau(D_1, D_2, D_3) = 0$.

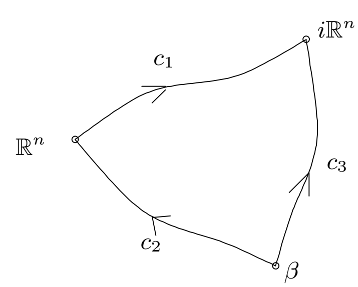

FIG. 8

Sinon $\tau(D_1, D_2, D_3)$ vaut 1 si D_1,D_2,D_3 sont dans cet ordre sur le cercle et -1 si elles sont dans l'ordre D_1,D_3,D_2.

(4.3) Revêtement universel du groupe symplectique. On a désigné par Γ le groupe des homéomorphismes de $\tilde{\Lambda}_n$ qui relèvent l'action d'un élément de $\mathrm{PSp}(2n, \mathbb{R})$ sur Λ_n. Il s'avère que, selon la parité de n, Γ est ou bien le revêtement universel du groupe symplectique, ou bien possède deux composantes connexes dont chacune est homéomorphe à un quotient par $\mathbb{Z}/2\mathbb{Z}$ de ce revêtement universel. Nous allons expliciter ce fait maintenant.

Tout d'abord Γ peut être identifié au sous-ensemble de $\mathrm{PSp}(2n, \mathbb{R}) \times \tilde{\Lambda}_n$ formé des couples $(g, \tilde{\alpha})$ tels que $g(*) = \pi(\tilde{\alpha})$. Ainsi on a un diagramme commutatif

$$\begin{array}{ccc} \Gamma & \longrightarrow & \tilde{\Lambda}_n \\ \downarrow & & \downarrow \pi \\ \mathrm{PSp}(2n, \mathbb{R}) & \longrightarrow & \Lambda_n \end{array}$$

et Γ revêt $\mathrm{PSp}(2n, \mathbb{R})$. Considérons de la même manière le revêtement $\widetilde{\Gamma}$ de $\mathrm{Sp}(2n, \mathbb{R})$ formé des couples $(g, \tilde{\alpha})$ de $\mathrm{Sp}(2n, \mathbb{R}) \times \tilde{\Lambda}_n$ tels que $g(*) = \pi(\tilde{\alpha})$.

$$\begin{array}{ccc} \widetilde{\Gamma} & \longrightarrow & \tilde{\Lambda}_n \\ \downarrow & & \downarrow \pi \\ \mathrm{Sp}(2n, \mathbb{R}) & \longrightarrow & \Lambda_n \end{array} .$$

L'application

$$\begin{aligned} \phi : \mathrm{Sp}(2n, \mathbb{R}) &\to \Lambda_n \\ g &\mapsto g(*) \end{aligned}$$

induit entre les groupes fondamentaux, après choix de deux générateurs, l'application $x \mapsto 2x$. On en déduit que $\widetilde{\Gamma}$ possède deux composantes connexes dont chacune est simplement connexe. On note donc $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ la composante de $(1, \tilde{*})$ dans $\widetilde{\Gamma}$ que l'on munit d'une structure de groupe en choisissant ce point comme élément neutre. C'est le revêtement universel du groupe symplectique.

Il nous reste maintenant à voir le lien entre $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ et Γ. On a bien sûr une application surjective $\varphi : \widetilde{\Gamma} \to \Gamma$ qui envoie $(g, \tilde{\alpha})$ sur $(\bar{g}, \tilde{\alpha})$, où $\bar{g}$ est la classe dans $\mathrm{PSp}(2n, \mathbb{R})$ de g. On a $\varphi(g_1, \tilde{\alpha}_1) = \varphi(g_2, \tilde{\alpha}_2)$ si et seulement si $\tilde{\alpha}_1 = \tilde{\alpha}_2$ et $g_1 = \pm g_2$. On doit donc voir quand les points $(g_1, \tilde{\alpha}_1)$ et $(-g_1, \tilde{\alpha}_1)$ sont dans la même composante connexe de $\widetilde{\Gamma}$. On vérifie facilement le lemme suivant.

Lemme (4.3.1). *Les éléments $(1, \tilde{*})$ et $(-1, \tilde{*})$ sont dans la même composante connexe de $\tilde{\Gamma}$ si et seulement si $(g, \tilde{\alpha})$ et $(-g, \tilde{\alpha})$ sont dans la même composante connexe pour tout $(g, \tilde{\alpha})$ de $\tilde{\Gamma}$.*

Considérons le chemin de Id à $-\mathrm{Id}$, $\gamma_t = e^{it}$ $(t \in [0, \pi])$ dans le groupe symplectique. Alors $\phi(\gamma_t)$ est un élément du groupe fondamental de la grassmannienne lagrangienne. On prouve facilement le lemme suivant :

Lemme (4.3.2). *Le lacet $\phi(\gamma_t)$ est dans l'image de ϕ_* si et seulement si $(-1, \tilde{*})$ est dans $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$.*

D'après le calcul du groupe fondamental de la lagrangienne effectué précédemment, un lacet de Λ_n est dans l'image de ϕ_* si et seulement si son image par l'application $\det^2$ parcourt un nombre pair de tours sur le cercle. Quand t parcourt $[0,\pi]$, $\det^2(\phi(\gamma_t))$ parcourt n tours sur le cercle. Donc :

– Si n est pair, $(1,\tilde{*})$ et $(-1,\tilde{*})$ sont dans la même composante connexe de $\widetilde{\Gamma}$. La restriction de φ à chacune des composantes de $\widetilde{\Gamma}$ est un revêtement à deux feuillets. Donc Γ a deux composantes connexes . Leur groupe fondamental est $\mathbb{Z}/2\mathbb{Z}$.

– Si n est impair, $(1,\tilde{*})$ et $(-1,\tilde{*})$ ne sont pas dans la même composante connexe de $\widetilde{\Gamma}$ et la restriction $\varphi : \widetilde{Sp}(2n,\mathbb{R}) \to \Gamma$ est un homéomorphisme. Le groupe des homéomorphismes de $\tilde{\Lambda}_n$ qui relèvent l'action du groupe symplectique sur la lagrangienne est bien le revêtement universel de $\mathrm{Sp}(2n,\mathbb{R})$.

Enfin on notera γ_T l'élément de $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ égal à $(1,T^2(\tilde{*}))$. La classe d'homotopie de lacets de $(1,\tilde{*})$ à γ_T dans $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ est le générateur du groupe du revêtement cyclique :

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \xrightarrow{p} \mathrm{Sp}(2n,\mathbb{R}) \longrightarrow 0\,.$$

(4.4) Classe de Maslov de $H^2(\mathrm{Sp}(2n,\mathbb{R})^\delta,\mathbb{R})$. On part maintenant d'une fonction $m_0 : \tilde{\Lambda}_n \times \tilde{\Lambda}_n \to \mathbb{Z}$ telle que :

$$m_0(\tilde{\alpha},\tilde{\beta}) + m_0(\tilde{\beta},\tilde{\gamma}) - m_0(\tilde{\alpha},\tilde{\gamma}) = \tau(\pi(\tilde{\alpha}),\pi(\tilde{\beta}),\pi(\tilde{\gamma}))\,.$$

Considérons la fonction

$$\begin{aligned}\phi &: \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \to \mathbb{Z}\\ g &\mapsto m_0(\tilde{*},\varphi(g)\tilde{*})\end{aligned}$$

Puisque $m_0(\tilde{\alpha},T(\tilde{\beta})) = m_0(\tilde{\alpha},\tilde{\beta}) + 2$ et que $\varphi(\gamma_T) = T^2$, la fonction ϕ satisfait : $\phi(g\gamma_T) = \phi(g) + 4$. Notons $c(g_1,g_2) = \phi(g_1) + \phi(g_2) - \phi(g_1g_2)$. Vu l'équation satisfaite par m_0 on a immédiatement : $c(g_1,g_2) = \tau(*,p(g_1)*,p(g_1g_2)*)$. La fonction c est donc bien définie sur le groupe symplectique $\mathrm{Sp}(2n,\mathbb{R})$ et y définit un 2-cocycle borné. La classe de cohomologie ainsi définie est par définition la classe de Maslov. Elle n'est pas triviale à priori puisque la fonction ϕ n'est pas définie sur le groupe symplectique. Elle peut être reliée à la classe de l'extension centrale :

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \xrightarrow{p} \mathrm{Sp}(2n,\mathbb{R}) \longrightarrow 0\,\cdot$$

En effet si l'on choisit une section ensembliste $s : \mathrm{Sp}(2n,\mathbb{R}) \to \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ (avec $s(1) = 1$) on peut écrire :

$$s(x)s(y) = \gamma_T^{n_{xy}} s(xy).$$

Alors la fonction $f\colon (x,y) \mapsto n_{xy}$ est un 2-cocycle qui représente dans $H^2(\mathrm{Sp}(2n,\mathbb{R})^\delta,\, \mathbb{Z})$ l'extension centrale que l'on considère. On a $\phi(s(x)s(y)) = \phi(s(xy)) + 4n_{xy}$, donc $c(x,y) + 4n_{xy}$ est le cobord de $-\phi\circ s$. Ainsi, $c + 4f$ est nul dans $H^2(\mathrm{Sp}(2n,\mathbb{R})^\delta,\mathbb{Z})$.

On peut également voir que cette construction ne dépend pas du point base $\tilde{*}$ choisi dans $\tilde{\Lambda}_n$. Le choix du point $\tilde{u}$ aurait conduit à une fonction ϕ_u définie de manière analogue à ϕ. Elle satisfait encore l'identité $\phi_u(g\gamma_T) = \phi_u(g) + 4$, donc la fonction $\phi - \phi_u$ est bien définie sur $\mathrm{Sp}(2n,\mathbb{R})$ et les cobords de ϕ et ϕ_u définissent la même classe de cohomologie réelle. On peut en fait dire mieux. Notons Φ l'unique quasi-morphisme homogène à distance bornée de ϕ : $\Phi(g) =$

$\lim_{p\to\infty} \frac{\phi(g^p)}{p}$. On définit de même Φ_u. Alors, puisque les deux quasi-morphismes homogènes Φ et Φ_u satisfont $\Phi(g\gamma_T) = \Phi(g) + 4$ ils sont égaux (cf. [3]). Donc la fonction $\phi - \phi_u = \phi - \Phi + (\Phi_u - \phi_u)$ est bornée et les cobords de ϕ et ϕ_u définissent la même classe dans le second groupe de cohomologie bornée de $\mathrm{Sp}(2n, \mathbb{R})$. La classe de Maslov peut donc être considérée comme une classe bornée. On pourra consulter [3] pour plus de détails sur ces faits.

Remarque. Dans [3], les auteurs obtiennent que c et f définissent les même classes de cohomologie. Nous obtenons $c = -4f$. Le signe opposé provient probablement d'un choix différent pour une orientation ou pour le générateur du groupe d'un revêtement. Expliquons d'où vient le 4. Dans [2], Arnold définit l'indice de Maslov de L_1,L_2,L_3 comme le nombre de valeurs propres positives ν_+ de la forme quadratique sur L_2, $x \mapsto \omega(p_1(x), p_3(x))$. Il ne considère que les triplets de lagrangiens deux à deux transverses. Il construit alors une primitive $\tilde{m} : \tilde{\Lambda}_n \times \tilde{\Lambda}_n \to \mathbb{Z}$ de cette fonction dans le cas des relevés de lagrangiens transverses. C'est cette primitive qui est utilisée dans [3]. Puisque dans les cas transverses, le rang de la forme quadratique est fixe égal à n, la signature $\nu_+ - \nu_-$ est aussi $2\nu_+ - n$. L'indice de Maslov comme nous l'avons défini est donc (à une constante près) le double de celui d'Arnold.

Le second facteur 2 provient du fait suivant. Dans [3] les auteurs affirment que le groupe Γ est le revêtement universel du groupe symplectique et que le générateur du noyau de $\widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$ est l'élément T qui engendre le groupe du revêtement $\tilde{\Lambda}_n \to \Lambda_n$. Nous avons vu qu'il existe en fait une application $\varphi : \widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \to \Gamma$ (qui n'est pas toujours un homéomorphisme) et que le générateur du noyau de $\widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$ s'envoie sur l'élément T^2 par φ, ce qui explique la présence du second facteur 2.

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Unité de Mathématiques Pures et Appliquées
École Normale Supérieure de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE
Pierre.Py@umpa.ens-lyon.fr

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